Generating Trees of (Reducible) 1324-avoiding Permutations*

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Abstract

We consider permutations that avoid the pattern 1324. We give exact formulas for the number of reducible 1324-avoiding permutations and the number of {1324, 4132, 2413, 3241}-avoiding permutations. By studying the generating tree for all 1324-avoiding permutations, we obtain a recurrence formula for their number. A computer program provides data for the number of 1324-avoiding permutations of length up to 20.

1 Introduction

Let S_n denote the set of all permutations of length n. A permutation $\pi = (p_1, p_2, \dots, p_n) \in S_n$ contains a pattern $\tau = (t_1, t_2, \dots, t_k) \in S_k$ if there is a sequence $1 \le i_{t_1} < i_{t_2} < \cdots i_{t_k} \le n$ such that $p_{i_1} < p_{i_2} < \cdots < p_{i_k}$. A permutation π avoids a pattern τ, in other words π is τ-avoiding, if π does not contain τ. We write $S_n(\tau)$ for the set of all τ-avoiding permutations of length n, and $s_n(\tau)$ for the cardinality of $S_n(\tau)$. Patterns τ_1 and τ_2 are *Wilf-equivalent* if $s_n(\tau_1) = s_n(\tau_2)$ [Wil02]. A permutation π is $\{\tau_1, \tau_2, \dots, \tau_n\}$-avoiding if π does not contain any of the patterns from the set.

It is a natural and easy-looking question to ask for the exact formula for $s_n(\tau)$. However, this problem turns out to be very difficult. Although a lot of results on this and related problems have been discovered in the last thirty years, exact answers are only known in a few cases. For all patterns τ of length 3, $s_n(\tau) = C_n$ [Knu73], where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number, a classical sequence [Sta99]. When τ is of length 4, it has been shown that the only essentially different patterns are 1234, 1342 and 1324; all other patterns of length 4 are Wilf-equivalent to one of these three [Sta94, Sta96, BW00]. Regev [Reg81] showed that $s_n(1234)$ asymptotically equals $c\frac{9^n}{n^4}$, where c is a constant given by a multiple integral. Gessel [Ges90] later used theory of symmetric functions to give a generating function for 1234-avoiding permutations. Bóna [Bón97a] enumerated 1342-avoiding permutations, giving their ordinary generating function:

$$\sum_n s_n(1342)x^n = \frac{32x}{-8x^2 + 20x + 1 - (1-8x)^{3/2}}.$$

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However, the exact enumeration of 1324-avoiding permutations is still an outstanding open problem that we address in this paper.

The problem of avoiding more than one pattern was first studied by Simion and Schmidt [SS85], who determined the number of permutations avoiding two or three patterns of length 3. The numbers of permutations avoiding certain pairs of patterns of length 4 give the Schröder numbers [Wes95]. West [Wes96] also used *generating trees* [CGHK78] to enumerate permutations avoiding all pairs of a pattern of length 3 and a pattern of length 4. Recently, Albert et al. [AAA+03] enumerated {1324, 31524}-avoiding permutations, while finding connections with queue jumping.

We present several results on enumerating 1324-avoiding permutations. Although the general problem still remains open, we enumerate some interesting subclasses, establishing yet another connection with Catalan and Fibonacci numbers. One subclass consists of *reducible* permutations; a permutation $(p_1, p_2, \dots, p_n) \in S_n$ is reducible[1] if there exists $1 \le i < n$, such that $\max_{1\le j\le i} p_j < \min_{i+1\le j\le n} p_j$. More importantly, we provide a full characterization for the generating tree of 1324-avoiding permutations. This result, combined with a simple computer program, provides data for $s_n(1324)$ for n up to 20. In particular, we show the following:

Theorem 1. *The number of reducible* 1324*-avoiding permutations of length* n *is* $C_{n+1} - 3C_n + 2C_{n-1}$*, where* C_n *denotes the* n*-th Catalan number.*

Theorem 2. *The number of {1324, 4132, 2413, 3241}-avoiding permutations of length* n *is* nF_{2n-3}*, where* F_j *denotes the* j*-th Fibonacci number.*

Theorem 3. *The number* $s_n(1324)$ *of all* 1324*-avoiding permutations of length* n *is* $g(\langle 1\rangle, n)$*, where* g *is determined by the following recursive formula:*

$$g(\langle a_1 \dots a_m\rangle, n) = \begin{cases} \sum_{i=1}^{m} a_i & \text{if } n = 1, \\ \sum_{i=1}^{m} g(f(\langle a_1 \dots a_m\rangle, i), n-1) & \text{if } n > 1 \end{cases} \tag{1}$$

and $f(\langle a_1 \dots a_m\rangle, i) = \langle b_1 \dots b_{a_i}\rangle$*, where:*

$$b_j = \begin{cases} a_i + 1 & \text{if } j = 1, \\ \min(i+1, a_j) & \text{if } 2 \le j \le i, \\ a_{j-1} + 1 & \text{if } i < j \le a_i. \end{cases} \tag{2}$$

The rest of this paper is organized as follows. Section 2 describes generating trees, the tool that we use in Section 3 to enumerate reducible 1324-avoiding permutations. In Section 3 we also enumerate {1324, 4132, 2413, 3241}-avoiding permutations. In Section 4 we characterize the generating tree of 1324-avoiding permutations. We conclude by enumerating 1324-avoiding permutations in a specific *strong* class, which is conjectured to be the largest. Finally, we present a conjecture regarding the growth of $s_n(1324)$.

[1] We previously used the term "separable", but it is already defined [BBL98, Knu73]. For more history on the term "reducible", see [Kla03, Com74]. Note that reducible permutations are dual to Bóna's decomposable permutations; in other words, reducible 1324-avoiding permutations are in an obvious bijection with decomposable 4231-avoiding permutations.

2 Generating trees

In this section we briefly describe succession rules and generating trees. They were introduced in [CGHK78] for the study of Baxter permutations and further applied to the study of pattern-avoiding permutations by Stankova and West [Sta94, Sta96, Wes95, Wes96]. Recently, Barcucci et al. developed ECO [BDLPP99], a methodology for the enumeration of combinatorial objects, which is based on the technique of generating trees.

Definition 4. A generating tree is a rooted, labelled tree such that the labels of the set of children of each node v can be determined from the label of v itself. In other words, a generating tree can be specified by a recursive definition consisting of:

1. **basis:** the label of the root
2. **inductive step:** a set of succession rules that yields a multiset of labelled children depending solely on the label of the parent.

Before we use generating trees for enumerating pattern-avoiding permutations, we introduce some more notation. Given $\pi = (p_1, p_2, \dots, p_n) \in S_n$, we call the position to the left of p_1 position 0, the position between p_i and p_{i+1}, where $1 \leq i \leq n-1$, position i, and the position to the right of p_n position n. We will refer to any of these positions as a *site* of π.

Definition 5. Let τ be a forbidden pattern. The position i, $0 \leq i \leq n$, of a permutation $\pi \in S_n(\tau)$ is an *active site* if inserting $n+1$ into position i gives a permutation belonging to the set $S_{n+1}(\tau)$; otherwise it is said to be an *inactive site*.

Example 6. The permutation $\pi = 13542 \in S_5(1324)$ has 4 active sites (the positions 0, 1, 2, and 3) and 2 inactive sites (the positions 4 and 5) as, e.g., $163542 \in S_6(1324)$ and $135462 \notin S_6(1324)$.

Following the methodology developed in [Wes96, Wes95], the generating tree for τ-avoiding permutations is a rooted tree whose nodes on level n are exactly the elements of $S_n(\tau)$. The children of a permutation π of length n are all the τ-avoiding permutations obtained by inserting $n+1$ into π. Each node in the tree is assigned a label; in the simplest case, the label is the number of active sites of π.

Example 7. The generating tree for 123-avoiding permutations (Figure 1) is given by the following:

$$\begin{cases} \text{basis:} & (2) \\ \text{inductive step:} & (k) \to (k+1)(2)\dots(k). \end{cases}$$

The permutation of length 1 has 2 active sites, which gives the basis rule. Let $\pi = (p_1 \dots p_n) \in S_n(123)$ and let k, $2 \leq k \leq n$, be the minimum index in π such that $p_i < p_k$ for some $i < k$. Then the active sites of π are the positions $0, 1, \dots, k-1$. Inserting $n+1$ into any other site to the right of the position $k-1$ results in a forbidden subsequence $(p_i, p_k, n+1)$. In other words, the active sites of π are the positions to the left of the end of the longest initial decreasing subsequence in π. The permutation obtained by inserting $n+1$ into the position 0 gives a new permutation with $k+1$ active sites; the permutation obtained by inserting $n+1$ into the position i, $1 \leq i \leq k-1$, has $i+1$ active sites. This gives the inductive step.

The number of 123-avoiding permutations of length n is thus the number of nodes on the n-th level of the above tree. It is easy to show [Wes95] that this number is C_n, the n-th Catalan number. Therefore, $s_n(123) = C_n$.

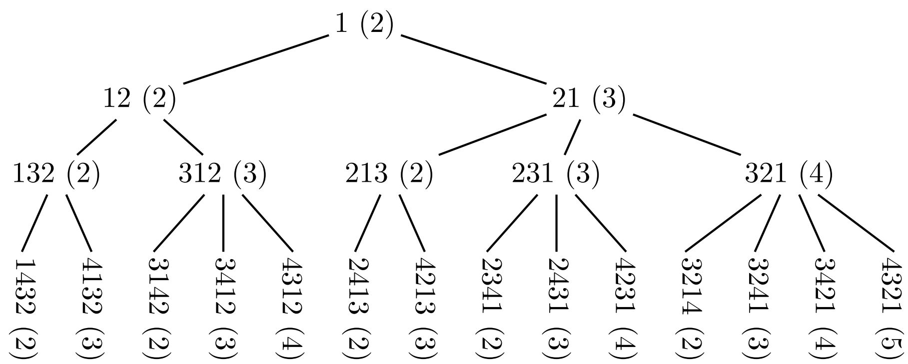

Figure 1: The generating tree for 123-avoiding permutations

3 Proofs of Theorems 1 and 2

Let a_n be the number of reducible 1324-avoiding permutations of length n. We apply generating trees to find a_n. First, we find b_n, the number of irreducible 132-avoiding permutations.

Lemma 8. *$b_0 = b_1 = 1$, and for all $n \geq 2$, $b_n = C_n - C_{n-1}$.*

Proof. It is known that $s_n(132) = C_n$. We prove that for $n \geq 2$, $b_n = s_n(132) - s_{n-1}(132)$ by showing that the number of reducible 132-avoiding permutations of length n is the same as the number of 132-avoiding permutation of length $n-1$.

Let A_n be the set of reducible 132-avoiding permutations of length n and let $B_n = S_{n-1}(132)$. We show a bijection from A_n to B_n. If $\pi \in A_n$, then $p_n = n$. Otherwise, if $p_n < n$ and π is 132-avoiding, then all the elements to the left of n are greater than all the elements to the right of n, and π would be irreducible. Therefore, by erasing n from π, we obtain a 132-avoiding permutation in S_{n-1}. If $\sigma \in B_n$, then inserting n as the last element generates a reducible 132-avoiding permutation. □

Lemma 9. *For all $n \geq 0$, $a_n = \sum_{k=1}^{n-1} b_k \cdot C_{n-k}$.*

Proof. We show that there are exactly two ways to obtain a 1324-avoiding reducible permutation of length n by inserting n into a permutation of length $n-1$: 1) insert n into an active site of a reducible 1324-avoiding permutation of length $n-1$ or 2) insert n at the end of an irreducible 132-avoiding permutation of length $n-1$.

Let σ_{n-1} be an irreducible 132-avoiding permutation of length $n-1$. After inserting n at the end of σ_{n-1}, we obtain σ_n, a reducible 1324-avoiding permutation of length n that has 2 active sites, at positions $n-1$ and n, i.e., right in front and right behind n. σ_n is reducible at exactly one position, $i = n-1$. All the other sites of σ_n are inactive, since inserting $n+1$ into any of them would create an irreducible permutation.

Let π_{n-1} be a reducible 1324-avoiding permutation of length $n-1$, with exactly k active sites. After inserting n at the i-th active site of π_{n-1}, we obtain π_n, a reducible 1324-avoiding permutation with $i+1$ active sites: the two sites right in front and right behind n, as well as $i-1$ active sites of π_{n-1} positioned to the left of n. The active sites of π_{n-1} positioned to the right of n become inactive in π_n. We can insert $n+1$ in π_n right in front or right behind n because it

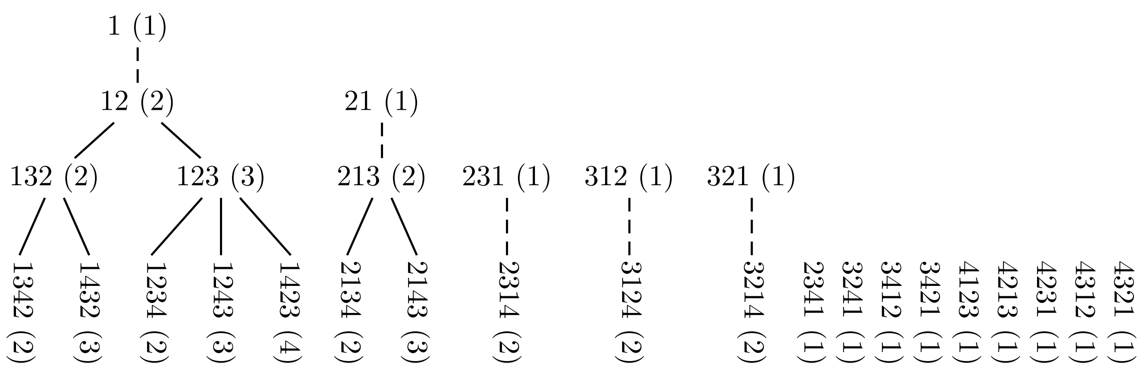

Figure 2: The generating forest for reducible 1324-avoiding permutations

does not create the forbidden 1324 pattern; otherwise, π_n would not have been 1324-avoiding, since n would have created 1324 with the same subsequence of type 132 and n playing the role of 4. Moreover, after inserting $n+1$ at one of these two sites, the permutation remains reducible at the same positions where it was reducible after inserting n. The $i-1$ active sites of π_{n-1} to the left of n remain active for the same reason: they cannot introduce the forbidden pattern 1324, since $n+1$ must play the role of 4 and inserting n at the i-th active site of π_{n-1} does not introduce any new pattern of type 132 to the left of the i-th active site. Moreover, after inserting $n+1$ at the i-th active site, the permutation remains reducible at the same positions where it was reducible after inserting n. The active sites of π_{n-1} positioned to the right of the inserted n become inactive because inserting $n+1$ in any of them would create the forbidden pattern 1324 with $n+1$ being a 4 and n being a 3; entries playing the roles of 1 and 2 exist because the permutation π_n is reducible at a position to the left of n.

This implies that all reducible 1324-avoiding permutations of length n lie on the n-th level of a generating forest (Figure 2) whose trees are rooted at an irreducible 132-avoiding permutation of length smaller than n and defined by the following succession rules:

$$\begin{cases} \text{basis:} & (1) \quad \text{an irreducible 132-avoiding permutation} \\ \text{inductive step:} & (1) \to (2) \\ & (k) \to (2)(3)\dots(k+1), \quad k \geq 2. \end{cases}$$

Example 7 shows that every generating tree in this forest is a Catalan tree; thus, the number of nodes at level[2] j is C_j. The total number of nodes at level n in this forest is $\sum_{i=1}^{n-1} b_i \cdot C_{n-i}$, because the nodes at level n in the forest correspond to the nodes at level $n-i$ in b_i trees. □

We next prove Theorem 1.

Proof. Using Lemma 9, we have that:

$$a_n = b_1 \cdot C_{n-1} + \sum_{i=2}^{n-1}(C_i - C_{i-1}) \cdot C_{n-i} = C_{n-1} + \sum_{i=2}^{n-1} C_i C_{n-i} - \sum_{i=2}^{n-1} C_{i-1} C_{n-i} = C_{n+1} - 3C_n + 2C_{n-1}.$$

[2]The root (1) has level 0.

The third equality follows from the recurrence for Catalan numbers: $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$. □

We next prove Theorem 2 that enumerates 1324-avoiding permutations that additionally avoid 4132, 2413, and 3241 patterns, i.e., all the *circular* variants of the 1324 pattern. This proof does not use generating trees.

Proof. Let d_n be the number of {1324, 4132, 2413, 3241}-avoiding permutations of length n. Let E_n be the set of {1324, 4132, 2413, 3241}-avoiding permutations π of length n such that $p_n = n$, and let e_n be its cardinality. Since 4132, 2413, and 3241 are the circular variants of the 1324 pattern, $d_n = ne_n$. Hence, it suffices to find e_n. Clearly, $e_1 = e_2 = 1$, $e_3 = 2$. Let $n \geq 4$. We consider i, the index of the entry $n-1$ in π. If $i \leq n-2$, then $\min\{p_1, \ldots, p_{i-1}\} > \max\{p_{i+1}, \ldots, p_n\}$; otherwise, there exists a 1324 pattern, where $n-1$ serves as 3 and n serves as 4. Therefore, $\{p_1, \ldots, p_i\} = \{n-i, n-i-1, \ldots, n-1\}$. Moreover, $p_1, \ldots, p_i$ appear in increasing order; otherwise there exists one of the remaining forbidden patterns with $n-1$ as one of its entries. Since any permutation satisfying these constraints on $p_1, \ldots, p_i$ is in E_n, we can delete the first i entries and obtain a trivial bijection with the permutations in E_{n-i}, counted by e_{n-i}. Finally, if $i = n-1$, that is, $p_{n-1} = n-1$, then deleting n, we obtain a bijection with the permutations in E_{n-1}, counted by e_{n-1}. Combining these two cases we obtain the recurrence relation:

$$e_n = e_{n-1} + \sum_{i=1}^{n-1} e_{n-i}$$

with the initial conditions $e_1 = e_2 = 1$. Now, it is just a matter of simple calculation to conclude that $e_n = F_{2n-3}$ and thus $d_n = nF_{2n-3}$. □

4 Proof of Theorem 3

In this section we apply generating trees to count *all* 1324-avoiding permutations. Typical applications of generating trees analyze changes in the number of active sites after inserting n in a permutation of length $n-1$. These changes determine the labels in the tree and the list of succession rules. Our application considers one more step: to keep the label of every node completely determined from the label of its parent, we consider the changes after inserting n and also $n+1$.

Given a node π at level $n-1$ in the generating tree for 1324-avoiding permutations, let π_n^i be π's children obtained by inserting n into the i-th active site of π. The label assigned to π_n^i is the pair $(s(\pi), i)$, where the sequence $s(\pi) = \langle l(\pi_n^1) \ldots l(\pi_n^{l(\pi)}) \rangle$ contains the number of active sites $l(\pi_n^j)$ for all children π_n^j of π, i.e., for π_n^i and all its siblings. The following completely characterizes this generating tree.

Lemma 10. *All* 1324*-avoiding permutations of length n lie on the n-th level of the generating tree (Figure 3) defined by the following succession rules:*

$$\begin{cases} \textit{basis:} & (\langle 2 \rangle, 1) \\ \textit{inductive step:} & (\langle a_1 \ldots a_m \rangle, i) \rightarrow (\langle b_1 \ldots b_{a_i} \rangle, a_i)(\langle b_1 \ldots b_{a_i} \rangle, a_i - 1) \ldots (\langle b_1 \ldots b_{a_i} \rangle, 1) \end{cases}$$

where $\langle b_1 \ldots b_{a_i} \rangle = f(\langle a_1 \ldots a_m \rangle, i)$ as in (2).

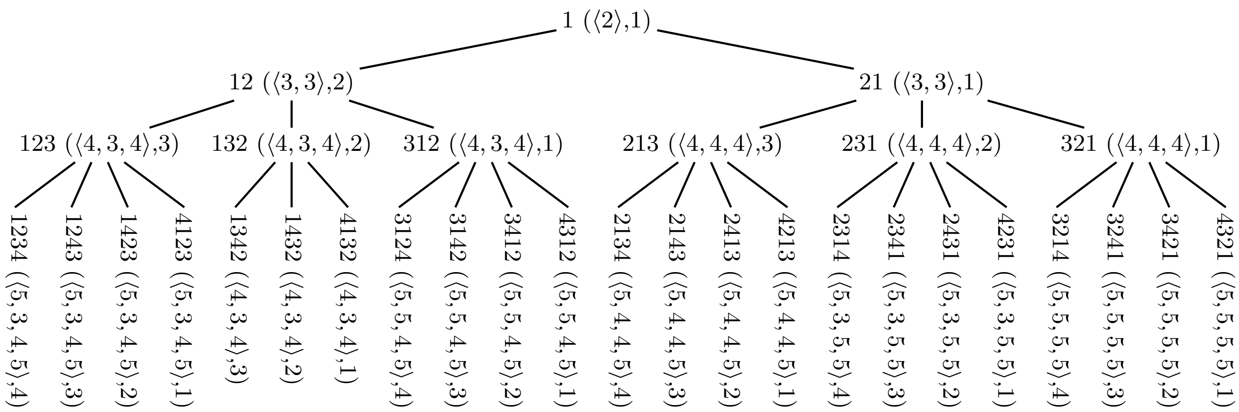

Figure 3: The generating tree for 1324-avoiding permutations

Proof. First, we make the following observation. Given a 1324-avoiding permutation $\pi = (p_1, p_2, \dots, p_{n-1})$ of length $n-1$, the active sites of π are actually the first $l(\pi)$ sites; we can order 132 patterns in π by the occurrence of their 2 and n can be inserted anywhere to the left of the first 2, but nowhere to the right of it.

Inserting n into the i-th active site of π certainly creates one new active site in π_n^i, since $n+1$ can be inserted into π_n^i right in front and right behind n. However, inserting n into π may deactivate some active sites in π, because n can play a role of 3 for some 132 pattern in π_n^i that was not in π. In other words, if we order 132 patterns in π and π_n^i by the occurrence of their 2, the first 2 in π_n^i may be to the left of the first 2 in π. The index of the first 2 that n introduces in π_n^i is $\min_{k>i-1, p_k>\min(p_1,p_2,\dots,p_{i-1})} k$. Since the active sites of π_n^i are exactly the sites to the left of the first 2, the number of active sites in π_n^i is:

$$l(\pi_n^i) = 1 + \min\{l(\pi), \min_{k>i-1, p_k>\min(p_1\dots p_{i-1})} k\} \tag{3}$$

Notice that $l(\pi_n^i) > i$, since $l(\pi) \geq i$ and $k \geq i$.

In the special case when $i = 1$, i.e., when π_n^i starts with n, we have $l(\pi_n^1) = 1 + l(\pi)$, since n cannot play the role of 3 for any 132 pattern. In general, however, the equation (3) does not express $l(\pi_n^i)$ solely in terms of $l(\pi)$. This is why we consider the next step, inserting $n+1$ into π_n^i.

Let $\pi_{n,n+1}^{i,j}$ be the permutation obtained by inserting $n+1$ into the j-th active site of π_n^i (which is not necessarily the j-th active site of π). We do a case analysis based on j; in each of three cases, the position of the first 2 is the key of our analysis:

- $j = 1$

  Then $\pi_{n,n+1}^{i,j}$ starts with $n+1$ and $l(\pi_{n,n+1}^{i,j}) = 1 + l(\pi_n^i)$.

- $2 \leq j \leq i$

  Then $n+1$ is inserted to the left of n and $\pi_{n,n+1}^{i,j} = (p_1, \dots, p_{j-1}, n+1, p_j, \dots, p_{i-1}, n, p_i, \dots, p_{n-1})$. Hence, $\pi_{n,n+1}^{i,j}$ has a 132 pattern where any element to the left of $n+1$ serves as 1, $n+1$ serves as 3, and n serves as 2. Thus, n may be the first 2 in $\pi_{n,n+1}^{i,j}$. Further, the number of active sites in $\pi_{n,n+1}^{i,j}$ equals the number of active sites in $\pi_n^j = (p_1, \dots, p_{j-1}, n, p_j, \dots, p_{n-1})$, unless n is the first 2 in $\pi_{n,n+1}^{i,j}$, which reduces the number of active sites in $\pi_{n,n+1}^{i,j}$ to the index of entry n. Therefore, $l(\pi_{n,n+1}^{i,j}) = \min(i+1, l(\pi_n^j))$.

- $i < j \leq l(\pi_n^i)$
  Then $n+1$ is inserted to the right of n and $\pi_{n,n+1}^{i,j} = (p_1, \dots, p_{i-1}, n, p_i, \dots, p_{j-2}, n+1, p_{j-1}, \dots, p_{n-1})$. Note that $n+1$ is inserted right behind p_{j-2}, and not p_{j-1}, because the position to the right of p_{j-2} is the j-th active site in π_n^i. The number of active sites in $\pi_{n,n+1}^{i,j}$ equals the number of active sites in $\pi_n^{j-1} = (p_1, \dots, p_{j-2}, n, p_{j-1}, \dots, p_{n-1})$ plus the additional active site next to entry n: $l(\pi_{n,n+1}^{i,j}) = l(\pi_n^{j-1}) + 1$.

In summary, we have obtained the number of active sites in a 1324-avoiding permutation of length $n+1$ in terms of the number of active sites in 1324-avoiding permutations of length n:

$$l(\pi_{n,n+1}^{i,j}) = \begin{cases} l(\pi_n^i) + 1 & \text{if } j = 1, \\ \min(i+1, l(\pi_n^j)) & \text{if } 2 \leq j \leq i, \\ l(\pi_n^{j-1}) + 1 & \text{if } i < j \leq l(\pi_n^i). \end{cases}$$

Clearly, the values $l(\pi_{n,n+1}^{i,j})$, $1 \leq j \leq l(\pi_n^i)$, depend on i and the values $l(\pi_n^j)$, $1 \leq j \leq l(\pi_n^i)$. Hence, if we assign label $(s(\pi), i)$, where $s(\pi) = \langle l(\pi_n^1) \dots l(\pi_n^{l(\pi)}) \rangle$, to each π_n^i, for $1 \leq i \leq l(\pi)$, then the label of $\pi_{n,n+1}^{i,j}$ is completely determined by the label of its parent, π_n^i. More precisely, the label of $\pi_{n,n+1}^{i,j}$ is $(s(\pi_n^i), j)$; the sequence $s(\pi_n^i) = \langle l(\pi_{n,n+1}^{i,1}) \dots l(\pi_{n,n+1}^{i,l(\pi_n^i)}) \rangle$ is given by the succession rule $s(\pi_n^i) = f(\langle l(\pi_n^1) \dots l(\pi_n^{l(\pi)}) \rangle, i)$, where f is the function defined in (2). The root of the tree has the label $(\langle 2 \rangle, 1)$, which represents the unique permutation of length 1. This completes the proof of the lemma. □

We next prove Theorem 3. Let T be the generating tree for 1324-avoiding permutations.

Proof. Let $d[(\langle a_1 \dots a_m \rangle, i), n]$ be the number of 1324-avoiding permutations on the n-th level of the subtree of T, rooted at (the node with label) $(\langle a_1 \dots a_m \rangle, i)$. Then,

$$d[(\langle a_1 \dots a_m \rangle, i), n] = \begin{cases} 1 & \text{if } n = 0, \\ \sum_{j=1}^{a_i} d[(\langle b_1 \dots b_{a_i} \rangle, j), n-1] & \text{if } n = 0. \end{cases}$$

Note that $d[(\langle a_1 \dots a_m \rangle, i), 1] = \sum_{j=1}^{a_i} d[(\langle b_1 \dots b_{a_i} \rangle, j), 0] = a_i$, since $d[(\langle b_1 \dots b_{a_i} \rangle, j), 0] = 1$.

Let $g(\langle a_1 \dots a_m \rangle, n)$ be the number of 1324-avoiding permutations on the n-th level of the subforest of T, which consists of trees whose roots are $(\langle a_1 \dots a_m \rangle, i)$, $1 \leq i \leq m$. Then,

$$\begin{aligned} g(\langle a_1 \dots a_m \rangle, n) &= \sum_{i=1}^{m} d[(\langle a_1 \dots a_m \rangle, i), n] = \sum_{i=1}^{m} \sum_{j=1}^{a_i} d[(f(\langle a_1 \dots a_m \rangle, i), j), n-1] \\ &= \sum_{i=1}^{m} g(f(\langle a_1 \dots a_m \rangle, i), n-1). \end{aligned}$$

□

```
count1324 := proc(n)
    return g([1], n);
end:

g := proc(s, n) option remember;
    local i, j, sum, sNext;
    if (n = 1) then
        return convert(s, `+`);
    fi;

    sum := 0;
    for i from 1 to nops(s) do
        sNext := s[i] + 1;
        for j from 2 to i do
            sNext := sNext, `min`(i + 1, s[j]);
        od;
        for j from i + 1 to s[i] do
            sNext := sNext, s[j - 1] + 1;
        od;
        sum := sum + g([sNext], n - 1);
    od;
    return sum;
end:
```

Figure 4: The Maple code for counting 1324-avoiding permutations

n	$s_n(1324)$
0	1
1	1
2	2
3	6
4	23
5	103
6	513
7	2,762
8	15,793
9	94,776
10	591,950
11	3,824,112
12	25,431,452
13	173,453,058
14	1,209,639,642
15	8,604,450,011
16	62,300,851,632
17	458,374,397,312
18	3,421,888,118,907
19	25,887,131,596,018
20	198,244,731,603,623

Figure 5: The number of 1324-avoiding permutations for length up to 20

5 Concluding remarks

Theorem 3 provides a recurrence formula for the number of 1324-avoiding permutations, which, with the help of a computer, gives values of $s_n(1324)$ up to $n = 20$ [SPBC96]. Figure 4 shows a simple Maple code that directly corresponds to Theorem 3; the procedure `count1324` counts the number of all 1324-avoiding permutations of length `n`, and the procedure `g` corresponds to g, with inlined f.

Note that `g` has `option remember` modifier. It instructs Maple to use memoization [Bel57, Mic68] for `g`. Namely, Maple maintains a table of the input pairs `s` and `n` and corresponding values for `g`. Before computing the value for some pair, Maple first checks if that pair is already in the table. If so, Maple immediately returns the value; otherwise, it computes the value and stores the pair and the value in the table. The use of memoization significantly reduces time for computing the values of `g` for larger `n`. However, the memoization table requires space. On machines on which we used Maple, it ran out of memory when `n` was 15. We rewrote the code from Figure 4 in Java to speed up the computation and to reduce the memory consumption. The Java code uses a more compact representation of sequences of small numbers. It also has a selective memoization that stores in the table only the input pairs (and their corresponding values) for which `g` is likely to be invoked several times. We ran the Java code on the Sun JVM version 1.3.0 running under Linux on a 2GHz Pentium IV machine with 2GB of memory. Computing the number of 1324-avoiding permutations of length 20 took about 5 hours.

Although we have obtained a recurrence formula for the number of all 1324-avoiding permutations, we do not have a closed form for $s_n(1324)$. The occurrence of the min function in the definition of f, together with the fact that the length of the sequences assigned to nodes of the generating tree increase with the node level in the tree, complicate any attempt to obtain a closed formula. But, the formula may help finding the asymptotic growth of $s_n(1324)$.

In 1990, Stanley and Wilf conjectured that $s_n(\tau) < (c(\tau))^n$, where $c(\tau)$ is a constant. This conjecture clearly holds for patterns of length 3. Results of Bóna and Regev [Bón97a, Reg81] imply that $s_n(1342) < 8^n$ and $s_n(1234) < 9^n$, these bounds being asymptotically tight. Moreover, Bóna [Bón97b] proves that $s_n(1324)$ is asymptotically larger than $s_n(1234)$, and that $s_n(1324) < 36^n$, the bound almost certainly not being tight. His idea for proving these two claims is elegant and simple; he considers permutations in strong classes, defined as follows.

Definition 11. Let $\pi \in S_n$. An element p_i is a left-to-right minimum if $p_i < p_j$, $\forall j \in [1, i-1]$. An element p_i is a right-to-left maxima if $p_i > p_j$, $\forall j \in [i+1, n]$.

Definition 12. Two permutations π and σ are said to be in the same *weak* class if the left-to-right minima of π are the same as those of σ and they are in the same positions. Moreover, π and σ are said to be in the same *strong* class if the above holds for their right-to-left maxima as well.

Example 13. The permutation 34125 has 2 left-to-right minima (1 and 3). The permutation 3612745 has 2 right-to-left maxima (7 and 5). $\{34125, 34152, 35124, 35142\}$ is a weak class, denoted[3] by $3 * 1 * *$, while 3612745 and 3416725 are the only 1324-avoiding permutations in the strong class $3 * 1 * 7 * 5$.

Bóna [Bón97b] shows that 1) every non-empty strong class contains a unique 1234-avoiding permutation and *at least* one 1324-avoiding permutation and 2) every strong class contains at most 4^n 1324-avoiding permutations. Combined with the fact that there are at most 9^n strong classes, this yields the upper bound of 36^n.

The values of $s_n(1324)$ in Figure 5 seem to suggest the following conjecture:

Conjecture 14. $s_n(1324) < 9^n$.

It is likely that Theorem 3 can be used to verify this conjecture, but we were not able to do so. Another approach is to try improving the 4^n bound on the number of 1324-avoiding permutations in any strong class. For example, Bóna [Bón97b] proved that a non-empty strong class, in which the right-to-left maxima occur next to each other in the rightmost positions, contains exactly one 1324-avoiding permutation.

Let $S_{l,r}$ denote the strong class in which l left-to-right minima occur in front of r right-to-left maxima, while the remaining entries are placed in the alternating positions. For example, $7 * 5 * 3 * 1 * 13 * 11 * 9$ is such a strong class with $l = 4$ and $r = 3$. Using the Java applet [Str03] provided by Atkinson and his group, we came up with the following interesting conjecture.

Conjecture 15. *The strong class $S_{l,r}$ contains more 1324-avoiding permutations than any other strong class with l left-to-right minima and r right-to-left maxima.*

We actually know the exact formula for $|S_{l,r}|$.

Proposition 16. $|S_{l,r}| = \binom{l+r-1}{l-1}$.

[3]We are using the notation from Bóna [Bón97b]. Note that both left-to-right minima and right-to-left maxima are decreasing (sub)sequences.

Proof. Let $n = 2k+1$. Let $a_l, \dots, a_1$ be the left-to-right minima, and $b_r, \dots, b_1$ be the right-to-left maxima. Here, the sequence $a_1, \dots, a_l, b_1, \dots, b_r$ is actually the sequence $1, 3, \dots, n$. Let $\sigma \in S_{l,r}$. It is easy to see that: 1) if $k+1$ occurs to the left of $b_r = n$, then $k+1$ has to be the second entry of σ; and 2) if $k+1$ occurs to the right of $a_1 = 1$, then $k+1$ has to be the next-to-last entry of σ. Hence, 1324-avoiding permutations in $S_{l,r}$ fall into two categories: the ones with $\sigma(2) = k+1$ and the ones with $\sigma(n-1) = k+1$. We map each $\sigma = (k, k+1, k-1, \gamma) \in S_{l,r}$ to $\sigma' = (k-1, \gamma') \in S_{l-1,r}$, and vice versa, where γ' is obtained from γ by reducing all the entries of γ that are greater than $k+1$ by 2. Therefore, 1324-avoiding permutations in $S_{l,r}$ with $k+1$ as the second entry are in one-to-one correspondence with 1324-avoiding permutations in $S_{l-1,r}$. Similarly, 1324-avoiding permutations in $S_{l,r}$ with $k+1$ as the next-to-last entry are in one-to-one correspondence with 1324-avoiding permutations in $S_{l,r-1}$. Thus, $|S_{l,r}| = |S_{l-1,r}| + |S_{l,r-1}|$, completing the proof by induction. □

Since $\binom{2r-1}{r-1} < 2^{n/2}$, the conjecture would prove that $s_n(1324) < (9\sqrt{2})^n$, which would be a considerable improvement on Bóna's bound.

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