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POWER SET MODELS OF LAMBDA-CALCULUS:

THEORIES, EXPANSIONS, ISOMORPHISMS

Giuseppe Longo

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Power Set Models of λ-Calculus: Theories, Expansions, Isomorphisms

by

Giuseppe Longo[1]

(1) I.S.I., Corso Italia 40, 56100 Pisa, Italy. Partially supported by a grant of the Italian C.N.R. to work at the Laboratory for Computer Science, Massachusetts Institute of Technology, Cambridge, MA, U.S.A.

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0-Introduction and summary: This paper mainly deals with the models for type free λ-calculus defined by Plotkin (Plo [1972]) and Engeler (Eng [1979]).

For every non empty set A, the model D_A is built up in a very natural set theoretic way and provides a code free generalization of early ideas of Scott, Scott [1976]. Namely, the notion of application (interpreting formal application of λ-terms) generalizes the classical Myhill-Shepherdson-Rogers definition of application in $P\omega$, introduced to define Enumeration Operators (see Ro [1967], p.143). Abstraction is defined accordingly.

An interesting fact is that these definitions do not depend on codings of pairs or of finite sets, while the classical ones do. This doesn't affect the Recursion Theory one should be able to work out on it, but does affect the model theory of λ-calculus (see BB [1979]). Moreover, for various reasons which should become clear in the next sections, Plotkin-Engeler's structures are very "handy", particularly in Engeler's version: it is easy to grasp the intuition on which the definitions and abstraction rely and to modify them for the purpose of the model theory of λ-calculus we aim at.

Section 1 (part I) introduces λ-terms and CL-terms (terms of λ-calculus, $\lambda\beta$, and of Combinatory Logic, CL) of various orders, corresponding to levels of functionality or number of λ-abstractions. Part II discusses the consequences in Combinatory Algebras of an early remark of Wadsworth (and Scott) on how to interpret the "loss of information" which is implicit in performing combinatory reductions, as in any effective process.

Section 2, following Barendregt's terminology (Bar [1981]), deals with the local analysis of Engeler's models, i.e. syntactically characterizes the true equalities in these models. Actually the partial order on these structures (i.e. set theoretic inclusion) matches perfectly well the very natural syntactical partial order over λ-terms, given by inclusion of Boehm-trees (the proofs are in Appendix B). This provides an algebraic characterization of λ-terms possessing normal form.

Section 3 gives a semantical characterization of λ-terms of any finite (and infinite) order, i.e., for $n \in \omega$, characterizes the class of terms such as $\lambda x_1 \ldots x_n.N$ according to n. In particular (closed) terms of order 0 are interpreted by the botton element of the lattice-theoretic model considered and terms of order infinity by the top element. This is done in a model â la Engeler, with a different interpretation of λ-abstraction.

Section 4 contains the model-theoretic applications of this paper. Theor. 4.1 summarizes what proved in the previous sections (different interpretation of λ-abstraction yield different sets of true equations). Theor. 4.5 deals with a purely algebraic consequence of the previous results. As already mentioned, Engeler's applicative structures generalize application as defined for enumeration reducibility in $\langle P\omega, \cdot \rangle$. In fact, $\langle D_A \cdot \rangle$ and $\langle P\omega, \cdot \rangle$ can be isomorphically embedded one into the other; but, using the previous local analysis, it is shown that for no A they are isomorphic (w.r. to " · ").

An Intermezzo and Appendix A discuss extensionality and "non well founded" models.

The notation is mainly from Bar [1981] and Mey [1981] unless explicitly defined (or elsewhere referred).

1. Approximation and Application.

Part I (Syntax): Let $CL\beta$ be CL extended with the axioms for strong reduction (Bar [1981], 7.3.6)

1.1 Def. (i) A CL-term M is of *order 0* ($M\epsilon 0_0$) iff $\neg\exists N\epsilon\{K,S,KU,SU,SVU/U,V \text{ CL-terms}\}\quad CL\beta \vdash M = N$

(ii) A CL-term is of *proper order 0* ($M\epsilon PO_0$) iff $M\epsilon 0_0$ and ($\neg\exists\vec{N}$ CL-terms $CL\beta \vdash M = x\vec{N}$).

(cf. C.F.[1958], p.145; $\vec{N},\vec{Q}$...are finite vectors (sets) of terms, possibly empty)

$CL\beta$ and $\lambda\beta$ are nicely related at a syntactical level. In particular on can go from λ-terms to CL-terms (and vice versa) preserving provable equalities (see Bar[1981], 7.1.4. - 7.3.1). Barendregt's translations $(\ )_\lambda: CL\rightarrow\lambda\beta$ and $(\ )_{CL}:\lambda\beta\rightarrow CL$, invertible up to provable equalities, are a tidy way for doing this.

Working in $\lambda\beta$, it is easy to define terms of order n, for any $n\epsilon\omega$, as well as terms of order infinity.

1.2 Def. Let M be a λ-term. Then

(i) $M\epsilon O_n$ iff n is the largest such that

$$\exists N \lambda\beta\vdash M=\lambda x_1...x_n.N.$$

(ii) $M\epsilon 0_\infty$ iff $\forall n\ M\notin 0_n$

Example. $YK\epsilon 0_\infty$, where Y is a "fixed point operator".

1.3 Proposition (i) Let M be a CL-term. Then

(i.0) $M\epsilon 0_0 \Leftrightarrow M_\lambda \epsilon 0_0$

(i.1) $M\epsilon PO_0 \Leftrightarrow M_\lambda\epsilon 0_0$ and $\neg\exists\vec{N}\ \lambda\beta \vdash M_\lambda = x\vec{N}$.

(ii) Let M be a λ-term. Then

$$M \in 0_\infty \quad \Leftrightarrow \quad \forall n\, \exists m{>}n\, \exists N\, \lambda\beta \vdash M = \lambda x_1 \dots x_m . N.$$

Proof. Easy. □

Clearly terms of order 0 are exactly the terms with no functionality.

(Un)solvable terms are defined in Bar[1981], p.41.

1.4 Lemma. M is unsolvable iff

(1) $M \in 0_\infty$, or

(2) $\exists n \geq 0\, \exists N \in PO_0 \quad \lambda\beta \vdash M = \lambda x_1 \dots x_m . N.$

Proof. $\Leftarrow$. By def. of head normal form (Bar[1981], P.41). $\Rightarrow$. We prove ($M \notin 0_\infty \Rightarrow$ (2)), when M is unsolvable. By 1.3.(ii), $M \notin 0_\infty \Leftrightarrow \exists n\, \forall m{>}n\, \forall N\, \lambda\beta \nvdash M = \lambda x_1 \dots x_m . N$.
If $n=0$, then $M \in PO_0$, since M is unsolvable.
If $n>0$, let n be the least such: then $\exists N \in 0_0\, \lambda\beta \vdash M = \lambda x_1 \dots x_n . N$. Actually $N \in PO_0$, for M is unsolvable. □

Part II (Semantics):

1.5 Def. Let " · " be a binary operation (application) on a set D. Then ⟨D, ·⟩ is a *Combinatory Algebra* iff D contains elements K and S satisfying $K \neq S$

$$K \cdot d_0 \cdot d_1 = d_0$$

$$S \cdot d_0 \cdot d_1 \cdot d_2 = d_0 \cdot d_2 (d_1 \cdot d_2) \text{ for all } d_0, d_1, d_2 \in D.$$

Thus in a Comb. Alg. ⟨D, ·⟩ one can interpret s and k of CL, by some S and K. For each choice of S and K in D, one obtains an *expansion* ⟨D, ·,S,K⟩ ⊨ CL, where CL contains s and k in the signature.

1.6 Def. $\langle D, \cdot, \Psi_\lambda \rangle$ is a *Combinatory Model* iff

(1) $\langle D, \cdot \rangle$ is a Comb. Alg.;

(2) Let $(D \to D) := \{f{:}D \to D / \exists d \in D\ \forall e \in D\ f(e) = d \cdot e\}$
then $\Psi_\lambda : (D \to D) \to D$;

(3) For $f \in (D \to D)$ and $e \in D$, $\Psi_\lambda(f) \cdot e = f(e)$.

By combinatory completeness, i.e. by 1.6(1), Comb. Models correspond to Environmental Models, as defined in HL[1980] or in Meyer[1981]. Meyer's Combinatory Model Theorem proves the equivalence of this notion with his purely algebraic definition of Comb. Model.

Let τ be an algebraic expression over D (see Bar[1981], p. 89; i.e. τ is built up with variables, constants from D and " $\cdot$ "). Let $\lambda x \in D.\tau$ be the function $d \vdash \tau[d/x]$. By combinatory completeness, i.e. by 1.6(1), $\lambda x \in D.\tau \in (D \to D))$, possibly with parameters. We write $\underline{\lambda}x.\tau$ for $\Psi_{\underline{\lambda}}(\lambda x \in D.\tau)$. So, for $f \in (D \to D)$, $\underline{\lambda}x.f(x)$ is the element of D which canonically represents the function f. Thus 1.6(3) reads $(\underline{\lambda}x.f(x)) \cdot e = f(e)$, which better recalls the schema (β) of λ-calculus (cf. Bar[1981], ch.5). By a small abuse of language, we will also write $\underline{\lambda}d.f(d)$ for $\underline{\lambda}x.f(x)$ and consider $\underline{\lambda}$ as a map from $(D \to D)$ into D, writing $\langle D, \cdot, \underline{\lambda} \rangle$ for $\langle D, \cdot, \Psi_{\underline{\lambda}} \rangle$.

Given $\langle D, \cdot \rangle$, there may be several choices of $\underline{\lambda}$; each one provides a specific λ-*expansion* of $\langle D, \cdot \rangle$. Each Combinatory Model $\langle D, \cdot, \underline{\lambda} \rangle$ naturally yields an expanded Comb. Alg.: set $\underline{K} = \underline{\lambda}xy.x := \underline{\lambda}x.(\underline{\lambda}y(x))$ $\quad \underline{S} = \underline{\lambda}xyz.xz(yz)$ (we omit " $\cdot$ " in $d \cdot e$). Following Mey[1981], we call these expanded Comb. Models λ-*Models.*

In view of Meyer's Lambda Model Theorem, we shall ignore the distinction between these interchangeable notions of model of $\lambda\beta$ and use the phrase λ-model throughout the rest of this paper (cf. also Bar[1981] or HL[1980]).

We write $D \models \ldots$ for $\langle D, \cdot, \underline{\lambda} \rangle$ ($\langle D, \cdot, \underline{S}, \underline{K} \rangle$) $\models \ldots$, if there is no ambiguity.

1.7 **Def.** Two (expanded) Comb. Alg. $\langle D_1, \cdot (,S_1,K_1)\rangle$ and $\langle D_2,x(,S_2,K_2)\rangle$ are *Equationally equivalent* iff $D_1 \models M=N \Leftrightarrow D_2 \models M=N$, for all CL-terms M,N.

As well known (β) or CL reductions entail a "loss of information". In $(\lambda x.M)N \rightarrow [N/x]M$, one knows "where one goes, but not where one comes from".

How can this be reflected in the semantics? Given a poset $\langle D,\leq\rangle$, let first say that $f:D\rightarrow D$ is *ω-continuous* iff, for any ω-chain $\{d_n\}_{n\in\omega}$, if $\sqcup d_n$ exists, then $f(\sqcup d_n) = \sqcup f(d_n)$.

Using ideas from Wadsworth's analysis of Scott's model D_∞, Wad[1976] (see also BL[1980], Bar[1981]) define:

1.8 **Def.** A Comb. Alg. $\langle D, \cdot\rangle$ has *approximable application*

iff (i) $\langle D, \cdot ,\leq\rangle$ is a poset, with least element $\perp$, such that " $\cdot$ ":$D^2 \rightarrow D$ is ω-continuous

(ii) There exists a map $Seg: D\times\omega \rightarrow D$ such that, for $d_n = Seg(d,n)$, $\forall d,e\in D$ one has

1- $d = \sqcup d_n$

2- $d_0 = \perp$

3- $\perp e = \perp$

4- $d_{n+1}e \leq (d\, e_n)_n$

5- $(d_n)_m = d_{min\{n,m\}}$

A way of understanding 1.8 may be the following:

- d_n is d up to "level n of information";

- applying no information ,$\perp$, to something, one gets no information;

- if the operator has level n+1 of information, then it uses at most level n of information from the input and provides at most an output with level n of information.

This has an immediate consequence for the semantics of the class of terms in CL where one can always perform reductions at the leftermost outermost level, i.e. for CL-terms in PO_0.

1.9 **Theorem** Let $\langle D, \cdot \rangle$ be a Comb. Alg. with approx. appl. . Then

$$M \in PO_0 \Rightarrow D \models M = \bot.$$

Proof. For the purpose of this proof, let's introduce a labeled CL, CL_0. The formation rules of CL-terms are extended by

$$M \in CL_0 \Rightarrow M^n \in CL_0, \text{ for all } n \in \omega;$$

the reduction rules are extended by

(Klab) 1. $K^{n+1} M \rightarrow (KM)^n$

2. $(KM)^{n+1} N \rightarrow M^n$

(Slab) 1. $S^{n+1} P \rightarrow (SP)^n$

2. $(SP)^{n+1} Q \rightarrow (SPQ)^n$

3. $(SPQ)^{n+1} R \rightarrow (PR^n(QR^n))^n$

(Min) $(M^n)^m \rightarrow M^{\min(n,m)}$

$M \in CL_0$ is *completely labeled* iff each occurrence of S and K in M is labeled. Interpret CL_0-terms in D, by adding $[\![M^n]\!]_\sigma = ([\![M]\!]_\sigma)_n$, for all environment σ.

__Claim 1__. Let $M \in PO_0$ and M^I a complete labeling of M. Then $CL_0 \vdash M^I \rightarrow N^0\vec{Q}$ for some $N, \vec{Q}$ in CL_0.

In fact by definition of PO_0-terms, K and S (labeled) rules are always applicable at the "head" of $M(M^I)$ and its contracts (in particular (Klab).2 and (Slab).3, up to label 0)

__Claim 2.__ If $CL_0 \vdash M \to N$, then $D \models M \leq N$

Use 1.8(ii) and monotonicity of " · ".

Let $M \in PO_0$. Then

$$D \models M = \sqcup\{M^I / I \text{ complete labeling}\} \quad \text{by 1.8(i)-(ii).1}$$

$$\leq \sqcup\{N^0\vec{Q} / \exists I \quad M^I \to N^0\vec{Q}\} \quad \text{by claim 1-2}$$

$$= \perp \quad \text{by 1.8.(ii).2-3, Q.E.D..}$$

So much for Combinatory Algebras; Theor. 1.9 in full generality will be applied in 3.6 and 4.5.

In the next sections we will use two notions of "tree of a λ-term". For the notion of Böhm-tree of a λ-term M, BT(M), we refer to Bar[1977] (or Bar[1981]). The partial order "$\subseteq$" on Böhm trees is the usual syntactic one: informally, put the always undefined element "$\perp$" at the bottom and then proceed inductively on the structure of the tree. Recall that $BT(M)=\perp$ iff M is unsolvable.

1.10 __Def.__ (Informal) Let $\Sigma = \{\lambda x_1...x_n.\perp / n\geq 0\} \cup \{T\} \cup \{\lambda x_1...x_n.y/n\geq 0\}$. Then the *Tree* of M, T(M), is a Σ-labelled tree defined as follows.

$$T(M) = T \quad \text{if } M \in 0_\infty,$$
$$T(M) = \lambda x_1...x_n.\perp \quad \text{if M is unsolvable of order n,}$$
$$T(M) = \lambda x_1...x_n.y$$

$$T(M_1).....T(M_p)$$

if M is solvable and has principal head normal form $\lambda x_1 \ldots x_n . y M_1 \ldots M_p$

A Tree may be infinite: just mimic Bar[1981; p.212] to give a formal definition. T(M) is obtained from BT(M) "displaying" the order of the unsolvable leaves. This can be done with the help of a Σ_2^0 oracle, writing his answers on leaves.

1.11 **Def.** The set of Trees is partially ordered by

$T(M) \subseteq T(N)$ iff some $\perp$'s in the leaves of T(M) are replaced by Trees of λ-terms <u>or</u> some $\lambda x_1 \ldots x_n . \perp$ are replaced by T.

<u>Example</u>

```
          λx.y                          λx.y
                        ⊆
    λz.⊥     λxy.⊥              λzv.x          T
                                      z
```

Given a λ-model $\langle D, \cdot, \underline{\lambda} \rangle$, embed $(D \to D)$ with the pointwise partial order.

1.12 **Theorem** Let $\langle D, \cdot, \underline{\lambda} \rangle$ be a λ-model with approximable application. Assume also that $\underline{\lambda}:(D \to D) \to D$ is monotone. Then

$$T(M) \subseteq T(N) \quad \Rightarrow \quad D \models M \subseteq N.$$

Proof. See Appendix C. □

An easy consequence of 1.12 is that, in a λ-model D as in 1.12, all fixed point operators Y of $\lambda\beta$ coincide in D and they represent Tarski's fixed point map $Y_T(f) = \sqcup f^n(\perp)$. See §3 for an application to a specific model.

1.13 **Def.** A λ-model $\langle D, \cdot, \lambda \rangle$ has λ-*approximable application* iff $\langle D, \cdot \rangle$ has approximable application and $\underline{\lambda} x . \perp = \perp$.

1.14 **Proposition.** Let $\langle D, \cdot, \underline{\lambda} \rangle$ be a λ-model with λ-approximable application. Then

$$BT(M) \subseteq BT(N) \Rightarrow D \models M \leq N.$$

Proof. Forcing $\lambda x.\bot = \bot$, $T(M)$ collapses to $BT(M)$: see appendix C for details. □

2. Engeler's Models and Their Local Structure

2.1 **Def.** Let $A \neq \emptyset$. Define

(i) $B_0 = A$
$B_{n+1} = B_n \cup \{(\beta;b)/\beta \text{ finite} \wedge \beta \subseteq B_n \wedge b \in B_n\}$
$B = \cup B_{(n)}$
$D_A = 2^B$

(ii) " · ":$D_A{}^2 \rightarrow D_A$ by $d \cdot e = \{b/\exists \beta \subseteq e\ (\beta;b) \in d\}$

(iii) $\Psi_{\underline{\lambda}}:(D_A \rightarrow D_a) \rightarrow D_A$ by
$\Psi_{\underline{\lambda}}(f) = \underline{\lambda}x.f(x) = \{(\beta;b)/b \in f(\beta)\}$

(Notation: α,β,γ... range over finite sets; we omit " · " in $d \cdot e$;
no element of A is denoted by (...;...))

This definition is due to Engeler (Eng[1979]; see also Plotkin[1972] and Scott[1980]).

2.2 **Lemma** For any $A \neq \emptyset$, $\langle D_A, \subseteq \rangle$ is a complete lattice. Scott's topology on D_A is given by the basis

$$\{\emptyset\} \cup \{\{d \in D_A/\beta \subseteq d\}/\beta \in D_A\}$$

.

Proof. Obvious □

2.3 **Lemma** Let $A \neq \emptyset$. Then the following are equivalent, over D_A:

1- $f:D_A \rightarrow D_A$ is continuous

2- $f(d) = \bigcup_{\beta \subseteq d} f(\beta)$

3- $(\underline{\lambda}x.f(x))d = f(d)$.

Proof. Routine □

Thus $\langle D_A, \cdot, \underline{\lambda} \rangle$ is a λ-model and $(D_A \rightarrow D_A) = C(D_A, D_A)$, the continuous functions (see also Eng[1979]). As usual $\underline{\lambda}x.dx := \underline{\lambda}x.f(x)$, for f represented by d.

The intuition on which the construction of D_A is based in clear : $(\beta;b)$ is an "elementary instruction" giving output b any time the input contains β. Thus $\forall d \in D_A$ $Ad=\emptyset$, since we assume A not to contain pairs such as $(\beta;b)$.

Note also that, by definition,

2.4 $\forall b \in B\ \exists \beta_n,\ldots,\beta_1\ \exists a \in A\ b = (\beta_n; \ldots.(\beta_1;a)\ldots)$.

This makes D_A "well founded" in the following sense: there is no infinite descending chain w.r.. to (the transitive closure of) the binary relation $\prec$ on B, where $b \prec (\beta;c) \Leftrightarrow b \in \beta \vee b=c \vee (c \in B \backslash A \wedge b \prec c)$.

The point is now to turn $\langle D_A, \cdot, \underline{\lambda} \rangle$ into a λ-model with λ-approximable application.

2.5 **<u>Def.</u>** (i) (Simultaneous definition of $|\cdot|$ on B and on the finite parts of B, with range in ω).

$$|b| = \begin{cases} 1 & \text{if } b \in A \\ |\beta|+|c| & \text{if } b=(\beta;c) \end{cases}$$

$$|\beta| = \max\{|c| / c \in \beta\}+1$$

(ii) Let $d \in D_A$. Define $d_n = \{b \in d / |b| \leq n\}$.
Clearly $|\beta|,|b| < |(\beta;b)|$.

2.6 **<u>Lemma</u>** $\langle D_A, \cdot, \underline{\lambda} \rangle$ has λ-approximable application.

<u>Proof</u> (*Part 1*: Approximable application) We only check 1.8(ii).4, the rest is trivial.

$$\begin{aligned} d_{n+1}e &= \{b / \exists \beta \subseteq e\ (\beta;b) \in d \wedge |\beta|+|b| \leq n+1\} \\ &\subseteq \{b / \exists \beta \subseteq e_n\ (\beta;b) \in d \wedge |b| \leq n\} = (de_n)_n, \end{aligned}$$

since $\forall \beta\ \forall b\ |\beta|,|b| \geq 1$ and $\max\{|c| / c \in \beta\} = |\beta|-1$.

(*Part 2*: λ-Approximable application) $\underline{\lambda}x.\emptyset = \{(\beta;b) / b \in \emptyset\} = \emptyset$. □

Thus 1.14 applies and $BT(M) \subseteq BT(N) \Rightarrow D_A \models M \subseteq N$.

To prove the reverse implication one can use the classical Böhm-out technique à la Hyland. A revised version of it is in BL[1980].[1] The point is to substitute Böhm's operator $c_p = \lambda x_0 ... x_{p+1}.x_{p+1}x_0...x_p$ by a C_p^- whose properties depend on the structure of D_A and such that Lemma 3.3 of BL[1980] applies. The construction of such a C_p^- required 26 technical lemmas, in the case of Plotkin's $T\omega$. For D_A it turns out to be much simpler and it is shown in Appendix B.

2.7 **Proposition** Let $A \neq \emptyset$. Then

$$BT(M) \subseteq BT(N) \Leftrightarrow \langle D_A, \cdot, \underline{\lambda} \rangle \models M \subseteq N.$$

Proof. By the proceeding remarks and Appendix B. □

Putting together 1.9 and 2.7 one has that M is unsolvable iff $\langle D_A, \cdot, \underline{\lambda} \rangle \models M = \emptyset$, thus $\langle D_A, \cdot, \underline{\lambda} \rangle$ is sensible in Barendregt's sense (Bar.[1981], p.100).

We conclude with a simple characterization of λ-terms possessing normal form.

Let $(D_A)^0 = \{[\![M]\!] / M \in \Lambda^0\}$ be the interior of D_A:$(D_A)^0$, as the set of objects interpreting closed λ-terms, can be algebraically characterized by taking

$\underline{S} = \underline{\lambda}xyz.xz(yz) \in D_A$ and $\underline{K} = \underline{\lambda}xy.x \in D_A$ and closing w.r.t. " · ".

2.8 **Corollary** Let $M \in \Lambda^0$. Then M has a normal form $\Leftrightarrow$ $\{d \in (D_A)^0 / D_A \models d \subseteq M\}$ is finite and $[\![M]\!]$ is maximal in $(D_A)^0$.

Proof. M has a normal forms iff BT(M) is finite and contains no Ω's. □

This fact is also true in the model $T\omega$; but the authors of BL[1980] were too distracted by the crazy hardware of $T\omega$, to point this out.

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[1] Correction for BL[1980], p.316, def. 3.4(ii), line 2: set $y_i = C_p^- \sigma_i$ instead of $\sigma_i = C_p^- y_i$, $0 \leq i \leq n$.

3. A Semantical Characterization of λ-terms of Order n, for any $n \in \omega \cup \{\infty\}$

.

In this section we define a different λ-expansion of the applicative structure $\langle D_A, \cdot \rangle$ defined in 2.1(i)-(ii). Namely, for each $f \in (D_A \to D_A)$, $\lambda^+ : (D_A \to D_A) \to D_A$ will choose a representative in the extensionality class of f, say $EC_f = \{d / \forall e\ f(e) = de\}$, different from $\underline{\lambda}(f)$.

3.1 **Def.** Let $A \neq \emptyset$, $\langle D_A, \cdot \rangle$ as in 2.1(i)-(ii) and $\underline{\lambda}$ as in 2.1(iii). Define $\lambda^+ : (D_A \to D_A) \to D_A$ by

$$\lambda^+(f) = \lambda^+ x.f(x) := \underline{\lambda} x.f(x) \cup A.$$

Note that for all $A \neq \emptyset$, the D_A's are objects of a Cartesian Closed Category (CPO's), with continuous maps as morphism. As already pointed out $(D_A \to D_A) = C(D_A, D_A)$: it is then easy to show that also $\underline{\lambda}$ and λ^+ are continuous maps. Moreover $C(D_A^n, D_A) \simeq C(D_A, C(D_A^{n-1}, D_A))$. Thus

$$\lambda^+ x_1 \ldots x_n.f(x_1, \ldots, x_n) = \lambda^+ x_1(\lambda^+ x_2 \ldots x_n.f(x_1, \ldots, x_n))$$

is well defined for all $f \in C(D_A^n, D_A)$. 3.3-5 show that $D_A^+ = \langle D_A, \cdot, \lambda^+ \rangle$ is a λ-model, for $A \neq \emptyset$ and $D_A, \cdot, \underline{\lambda}$ and λ^+ defined as in 3.1.

3.2 **Def.** Define $A_{(n)} = \lambda^+ x_1 \ldots x_n.\emptyset$.

3.3 **Lemma**

(i) $A_{(0)} = \emptyset$

(ii) $A_{(n+1)} = \lambda^+ x.A_{(n)} = \underline{\lambda} x_1, \ldots x_n.A \cup \underline{\lambda} x_1 \ldots x_{n-1}.A \cup \ldots \cup A$

(iii) $A_{(n)} \subset A_{(n+1)}$

(iv) $\forall d \in D_A\ (A_{(n+1)})d = A_{(n)}$

(v) $\lambda^+ x_1 \ldots x_n.f(x_1, \ldots, x_n) = \underline{\lambda} x_1 \ldots x_n.f(x_1, \ldots, x_n) \cup A_n$, for all $f \in C(D_A^n, D_A)$.

(vi) $\lambda^+ x_1 \ldots x_n . f(x_1,\ldots,x_n)\vec{d} = \lambda^+ x_{p+1} \ldots x_n . f(d_1,\ldots,d_p,\ldots,x_n)$,
for all $f \in C(D^n_A, D_A)$ and $\vec{d} = \{d_1,\ldots,d_p\}$, with $p \leq n$.

Proof. (i) Obvious; (ii) easy induction; (iii) by (ii).
(iv) by $Ad = \emptyset$, for all $d \in D_A$, and continuity of " · " (recall that $\underline{\lambda} x.A = \{(\beta;b)/b \in A\}$).
(v) induction, again; (vi) by (iv) and (v). □

3.4 **Lemma**. $\forall f \in (D_A \to D_A)$, $\lambda^+ x.f(x)$ is the largest element in EC_f.

Proof. Let $d \in EC_f$. Clearly $d \cap A \subseteq \lambda^+ x.f(x)$. Let $(\beta;b) \in d$. Then $b \in d\beta$, i.e. $b \in f(\beta)$ and we are done. □

3.5 **Corollary** $D^+_A = \langle D_A, \cdot, \lambda^+ \rangle$ is a λ-model. Moreover it is the unique λ-expansion of $\langle D_A, \cdot \rangle$ satisfying $\forall d \in D_A \quad d \subseteq \lambda^+ x.dx$.

Proof The first part is by 3.3; note that if $\lambda\beta \vdash M = \lambda x_1 \ldots x_n . N$, then

(1) $\forall \sigma \; [\![M]\!]^+_\sigma = \underline{\lambda} d_1 \ldots d_n . [\![N]\!]^+ \sigma[\vec{d}/\vec{x}] \cup A_{(n)}$, by 3.3(v).

Assume now that $\langle D_1, \cdot, \lambda' \rangle$ is a λ-expansion such that $\forall d \; d \subseteq \lambda' x.dx$. Recall that

(2) $\forall d,e \; (d \cup A)e = de$. Then, for all $f \in (D_A \to D_A)$,

$\underline{\lambda} y.f(y) \cup A$	$\subseteq$	$\lambda' x.(\underline{\lambda} y.f(y) \cup A)x$	by assumption
	$=$	$\lambda' x.(\underline{\lambda} y.f(y))x$	by (2)
	$=$	$\lambda' x.f(x)$	by 2.3

By 3.4, we are done. □

Clearly $\langle D_A, \cdot \rangle$ possess Card (2^A) λ-expansions. This λ-model provides a semantical characterization of the λ-terms in PO_0, 0_∞ and 0_n, for all n.

3.6 **Theorem** Let M be a λ-term. Then

$$M \in PO_0 \Leftrightarrow D^+_A \models M = \emptyset$$

Proof. $\Rightarrow$. This follows from 1.9. Notice that 1.9 depends only on the applicative structure of $\langle D_A, \cdot \rangle$, i.e. on $\langle D_A, \cdot \rangle$ as a Comb. Alg., not on the λ-expansions which may turn it into a λ-model.

$\Leftarrow$. Assume $M \notin PO_0$.

Case $\lambda\beta \vdash M = x\vec{Q}$, for some $Q_1, \ldots, Q_p$. Then, since $D_A^+ \models M = \emptyset \Leftrightarrow \forall\sigma [\![M]\!]_\sigma^0 = \emptyset$, $D_A^+ \not\models M = \emptyset$, by taking $\sigma(x) = \underline{\lambda} x_1 \ldots x_p.A$.

Case $\lambda\beta \vdash M = \lambda x.N$, for some N. Then

$$\forall\sigma \qquad [\![M]\!]_\sigma^+ = \lambda^+ d.[\![N]\!]^+ \sigma_x^d = \underline{\lambda} d.[\![N]\!]^+ \sigma_x^d \cup A \neq \emptyset. \qquad \square$$

Of course, 3.6. $\Leftarrow$. depends on the λ-expansion (cf. 2.7)

3.7 **Theorem** Let M be a λ-term. Then

(i) $M \in 0_n \Leftrightarrow (D_A^+ \models M \supseteq A_m \underset{(1)}{\Leftrightarrow} m \leq n)$

(ii) $M \in 0_\infty \Leftrightarrow D_A^+ \models M = B$

Proof. (i). $\Rightarrow$. Assume $\lambda\beta \vdash M = \lambda x_1 \ldots x_n.N$, with $N \in 0_0$. Then (1), in 3.5, immediately gives $\underset{(1)}{\Leftarrow}$.

As for $\underset{(1)}{\Rightarrow}$, assume m>n.

Case $N \in P0_0$. By 3.6 and (1), $\forall\sigma$ $[\![M]\!]_\sigma^+ = A_n$ (recall that $\underline{\lambda} x.\emptyset = \emptyset$), while $A_n \subset A_m$, for m>n.

Case $N \equiv x_i\vec{Q}$, for some $\vec{Q}$ and $i \leq n$. Take $b = (\beta_1; \ldots (\underset{i\text{-th}}{\emptyset;} \ldots (\beta_m; a) \ldots))$ for some $\vec{\beta}$ in D_A and $a \in A$. Clearly $b \in A_m \setminus A_n$; thus $b \in \underline{\lambda} d_1 \ldots d_n.d_i([\![\vec{Q}]\!]^+ \sigma [\vec{d}/\vec{x}])$. Then by the definition of $\underline{\lambda}$, one has $a \in \emptyset([\![\vec{Q}]\!]\sigma [\vec{\beta}/\vec{x}]) = \emptyset$.

Case $N \equiv y\vec{Q}$, for some $\vec{Q}$ and $y \neq x_i$, $i \leq n$. Take $\sigma(y) = \emptyset$, then by (1) $[\![M]\!]_\sigma^+ = A_n \subset A_m$, for m>n.

So far for (i). $\Rightarrow$.

$\Leftarrow$. Assume $M \notin 0_n$

Case $M \in 0_p$, with $p \neq n$. Then by part ,$\Rightarrow$, we are done.

Case $M \in 0_\infty$, Then, since $A_0 \subseteq A_1 \subseteq \ldots A_n \subseteq \ldots$,

(3) $\forall_p \forall_\sigma [\![M]\!]_\sigma^+ \supseteq A_p$, by 1.3(ii) and (1). This contradicts $\underset{(1)}{\Rightarrow}$, again.

(ii) Clearly $\cup A_{(n)} \subseteq B$. Conversely, if $b \in B = \cup B_n$ (see def. 2.1), $b = (\beta_1; \ldots(\beta_p; a))$ for some $\vec{\beta}$ in D_A, $a \in A$(by 2.4). By 3.3(ii) and 2.1(iii), $A_{(p+1)} \supseteq \underline{\lambda} x_1 \ldots x_p.A = \{(\beta_1; \ldots(\beta_p; a))/a \in A, \vec{\beta} \text{ in } D_A\}$.

Thus,

(4) $B = \cup A_{(n)}$.

Now, assume $M \in 0_\infty$, then

$\forall \sigma\ [\![M]\!]_\sigma^+ = B$, by (3) and (4).

Conversely, $D_A^+ \models M = B$ implies

$\forall n\ M \notin 0_n$, by (i) and (4). □

In view of (4), let's write $A_\infty = B$.

3.8 <u>Corollary</u> Let M be a λ-term. Then

M is unsolvable $\Leftrightarrow \exists n \in \omega \cup \{\infty\} \quad D_A^+ \models M = A_{(n)}$.

<u>Proof.</u> $\Rightarrow$. By 1.4 we have two cases.

Case $M \in 0_\infty$. Then $D_A^+ \models M = A_{(\infty)}$, by 3.7(ii).

Case $\lambda\beta \vdash M = \lambda x_1 \ldots x_n.N$, for some $n \in \omega$ and $N \in P0_0$.

Then $\forall \sigma [\![M]\!]_\sigma^+ = \underline{\lambda} \vec{x}.\varnothing \cup A_{(n)} = A_{(n)}$ by (1) and 3.6.

$\Leftarrow$. Assume $\lambda\beta\vdash M=\lambda x_1...x_m.yQ_1...Q_p$, for some $m\in\omega$, y and $\vec{Q}$ (i.e. assume that M is solvable).

Then, by 3.7(i),

$$D_A^+ \models M = A_{(n)} \Rightarrow n\leq m.$$

Thus, let $n\leq m$.

Case $y\neq x_i$, $\forall i\leq m$. Take $\sigma(y)=\underline{\lambda}x_1...x_p.A$.

Then $[\![M]\!]_\sigma^+ = \lambda^+ x_1...x_m.A$

$\neq A_{(n)}$, by 3.2-3.3.

Case $y=x_i$, for some $i\leq m$. Take $\beta_i^a = \{(\emptyset;...(\emptyset;a)..)\}$ of length p, for some $a\in A$, and $b^a = (\beta_1;...(\beta_i^a;...(\beta_m;a)...)$, for some $\vec{\beta}$.

Then $b^a \notin A_{(n)}$, by 3.3(ii) and $n\leq m$. Nonetheless

$$\forall\sigma \quad b^a\in\{(\beta_1;...(\beta_m;b)...)/b\in\beta_i[\![\vec{Q}]\!]^+\sigma[\vec{\beta}/\vec{x}]\} \subseteq [\![M]\!]_\sigma^+$$

since $\beta_i^a d_1...d_p=\{a\}$ and we are done. □

By 3.7, the witness n of the RHS of 3.8 is unique and it is the order of the unsolvable term M.

Note that D_A^+ provides a semantical characterization of unsolvable terms, with their functionality. Moreover the functionality of solvable terms is also characterized, by 3.7(i), though it never occurs that a solvable and an unsolvable term are equated. Finally, by the monotonicity of λ^+, 1.12 applies; thus $T(M) \subseteq T(N) \Rightarrow D_A^+ \models M \subseteq N$. The author believes that this model is "very sensible" although Henk Barendregt wouldn't call it so (cf. Bar[1981], p.100).

With some patience, one should also be able to work out the following fact:

$$T(M)=T(N) \Leftrightarrow D_A^+\models M=N$$

Intermezzo (on extensionality). Part 1 None of the models studied so far is extensional; namely, in general, EC_f contains more than one element. Throwing away some elements, can we turn $\langle D_A, \cdot \rangle$ into an extensional λ-model? There doesn't seem to be an elementary direct way for such a construction, starting from a λ-expansion of $\langle D_A, \cdot \rangle$. (see 4.4 for an indirect argument).

Scott[1976] (see also Scott[1980]) presents an elegant technique to construct an extensional substructure of the λ-model $P\omega$. This technique applies to "almost" (see later) any λ-model satisfying (η^-) $\lambda x.x \leq \lambda xy.xy$, which is a c.p.o..

Scott's argument is the following:

Let $I_0 := I \equiv \lambda x.x \qquad I_{n+1} \equiv \lambda xy.I_n(x(I_n y))$

Set $d_{(n)} = [\![I_n]\!] \in P\omega$ and $d_\infty = \cup d_{(n)}$. Then $E\omega = \{d_\infty e / e \in P\omega\} \subseteq P\omega$ is an extensional λ-model (see Scott[1980]).

Remark (i) Scott ([1980], p.251) points out that d_∞ is the least solution of

$$d = d_{(0)} \cup (\underline{\lambda} xy \cdot d(x(dy))),$$

and remarks that d_∞ doesn't seem to be the interpretation of any closed λ-term. And in fact it is not. By induction, one has

(1) $\forall n\ \lambda\beta \vdash I_n = \lambda x_0 \ldots x_n . x_0 (I_{n-1} x_1)(I_{n-2} x_2) \ldots x_n .$

now $d_\infty \neq \varnothing$; thus, if $[\![M]\!] = d_\infty$, M is solvable, say $\lambda\beta \vdash M = \lambda x_1 \ldots x_p . x_j \vec{Q}$. Then one can derive a contradiction from $\forall n\ d_n \subseteq [\![M]\!]$ and (1), by applying them to the right $\vec{C}$ in $P\omega$, depending on j.

(ii) Let $(B, \leq_\eta)$ be the set of Böhm-like trees partially ordered by (possibly infinite) η-expansions, see Bar[1981], p.230. It is then easy to show (use cofinality of chains) that $\sqcup BT(I_n)$ is the "Nakajima-tree" of $\lambda x.x$ (see Bar[1981], p.511). *Conjecture*: $E\omega$ is equationally equivalent to Scott's inverse limit D_∞.

Part 2. Scott's (η^-) property seems to be a fairly natural property for a λ-model $\langle D, \cdot, \lambda' \rangle$ which is a poset. It says that λ' chooses the largest element in EC_f, for each $f \in (D \rightarrow D)$. That is exactly what λ^+ does in the case of D_A^+. Nonetheless the technique of Part 1 doesn't apply to D_A^+ (thus to no other λ-expansion of $\langle D_A, \cdot \rangle$). In fact, by 3.7(i) and (4) in 3.7, one has $d_\infty = B$. Therefore

$$\{d_\infty e / e \in D_A\} = \{B\},$$

since $\forall e \in D_A \; Be = B$, i.e. the extensional substructure collapses to a singleton.

How to force (η^-) into Engeler's construction and still obtain an interesting d_∞?

Given $\lambda' \neq \lambda^+$, $\forall d \; d \subseteq \lambda' x.dx$ is false because of those d's containing elements of A, which do not act as "instructions" (see 3.5). Thus, what one can do, with $a \in A$, is to force $a = (\{a\};a)$ (or, also, $a = (\emptyset;a)$, see A.6), i.e. force a to act. That is, set

$$a \simeq (\{a\};a) \text{ and consider } \tilde{B} = B/\simeq, \; \tilde{D}_A = \langle 2^{\tilde{B}}, \cdot \rangle^1 .$$

$\tilde{D}_A$ is no longer well founded and there is no way of turning $\langle \tilde{D}_A, \cdot \rangle$ into a Comb. Algebra with approx. appl. : this is an immediate consequence of 1.9 and Cor. A.5 (appendix A), which gives a semantical characterization of closed λ-terms of order 0, different from 3.6; namely,

$$M \in 0_0 \Leftrightarrow \tilde{D}_A \models M = A, \text{ for M closed.}$$

Notice that, in $\tilde{D}_A$, $\forall b \in \tilde{B} \; (\emptyset;b) \notin d_\infty = \cup d_{(n)}$, by (1) in the remark. Thus $d_\infty \emptyset = \emptyset$; since one also has $d_\infty B = B$, then Scott's technique applies and the extensional substructure is not trivial.

In this model, Tarski's fixed point operator is not λ-definable (see A.5).

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[1] By a different argument, Scott derives the same observation (see remark on p.251, Scott[1980]

4. D_A and $P\omega$

The local analysis of $\langle D_A, \cdot, \lambda\rangle$ and of $\langle D_A, \cdot, \lambda^+\rangle$ has the following immediate consequence.

4.1 **Theorem** Let $A \neq \emptyset$. Then the Comb. alg. $\langle D_A, \cdot\rangle$ has λ-expansions which yield non equationally equivalent λ-models.

Another application of the previous results (namely 1.9 and 2.7) relates D_A and $P\omega$. In particular the isomorphisms between $\langle D_A, \cdot\rangle$ and $\langle P\omega, \cdot\rangle$ as applicative structures.

" $\cdot$ " over $P\omega$ is defined as for Enumeration Reducibility (see Ro[1967], p.146; Bar[1981], p.469). That is, for codings of the finite sets $\{E_n\}_{n\in\omega}$ and of pairs (,), $CG=\{m/\exists E_n \subseteq G\ (n,m)\in C\}$. By $\langle D,x\rangle \Rightarrow \langle D', \cdot\rangle$ we mean that D can be isomorphically embedded into D', w.r. to "x" and " $\cdot$ ".

4.2 **Proposition** Let $A \neq \emptyset$. Then one has

(i) $\langle P\omega, \cdot\rangle \Rightarrow \langle D_A, \cdot\rangle$
(ii) If A is countable, then $\langle D_A, \cdot\rangle \Rightarrow \langle P\omega, \cdot\rangle$.

Proof. (Notation: (,) and $\{E_n\}_{n\in\omega}$, finite sets, are as in Ro[1967]; in particular $E_0=\emptyset$, $E_1=\{0\}$. Set also $\#E_n=n$).

Notice first that

(0) $\forall n\in\omega\backslash\{0\}\ \exists!k\exists!n_1....\exists!r_k\ n=(n_1,...(n_k,0)...)\wedge n_k\neq 0.$

(i) Define (simultaneously), for some $a\in A$,

$[\cdot]:\omega\rightarrow D_A$
$h:\{E_n\}_{n\in\omega}\rightarrow\{\beta/\beta\subseteq B \text{ finite}\}$
$\text{first}: [\omega]\rightarrow B$

by

$[0]=\{a, (\emptyset;a), (\emptyset;(\emptyset;a)),......\}$,
$\text{first}([0])=a$;

Let $n=(n_1,(n_2,...(n_k,0)...),n_k\neq 0$: then set

$$[n]=\{(\beta'_{n_1};(\beta'_{n_2};...(\beta'_{n_k};b)...)/b\in[0]\}$$

where, for $E_n=\{m_1,...,m_q\}$,

$$\beta'_n:=h(E_n)=\{\text{first}([m_1]),...,\text{first}([m_q])\}\ (\text{with } h(\emptyset)=\emptyset)$$

and, for $p=(p_1,(p_2,...(p_\tau,0)\neq 0$,
$\text{first}([p])=(\beta'_{p_1};(\beta'_{p_2};...(\beta'_{p_\tau};a)...)$.

Finally define $f:P\omega\rightarrow D_A$ by

$$f(C)=\cup\{[n]/n\in C\}.$$

<u>Claim</u> (1) $(\beta'_m;b)\in[(n,p)]\Leftrightarrow m=n\wedge b\in[p]$.

(2) $E_n\subseteq C\in P\omega\leftrightarrow\beta'_n\subseteq f(C)$.

Part (1) easily follows by the definitions (note that it holds also for $(n,p)=0$, i.e. $n=0\wedge p=0$)

As for (2), notice that $\beta'_n \subseteq f(E_n)$. Clearly f is injective.

Compute now

$$f(C)f(G) = \{b / \exists \beta \subseteq f(G)\ (\beta;b) \in f(C)\}$$

$$= \{b / \exists n(\beta'_n \subseteq f(G) \wedge \exists p\ (\beta'_n;b) \in [(n,p)] \wedge (n,p) \in C)\} \quad \text{, by (1)}$$

$$= \{b / \exists E_n \subseteq G\ \exists p\ b \in [p] \wedge (n,p) \in C\} \quad \text{, by (2), (1)}$$

$$= \cup \{[p] / \exists E_n \subseteq G\ (n,p) \in C\}$$

$$= f(CG).$$

(ii) Define, for $A = \{a_0, a_1, a_2, \ldots\}$

$$b \to \underline{b} : B \to \omega$$
$$g : \{\beta / \beta \subseteq B(\text{finite})\} \to \omega$$

by

$$\underline{a_n} = (1,n)$$
$$\underline{\beta \to b} = (g(\beta), \underline{b})$$

where, for $\beta = \{b_1, \ldots, b_q\}$,

$$g(\beta) = \#\{\underline{b}_1, \ldots, \underline{b}_q\}$$

Define

$$f' : D_A \to P\omega \text{ by } f'(d) = \{\underline{b} / b \in d\}$$

<u>Claim</u> (1) $b \to \underline{b}$ is injective; g is injective

(2) $\forall d \in D_A\ 0 \notin f'(d)$

(3) f' is injective

As for (1)+(2), define $| \quad |: B \to \omega$ as in 2.5.

The proof easily follows by (combined) induction on $|b|$ (and $|\beta|$).

As for (3), it is an obvious consequence of (1).

Then compute

$$f'(d)f'(e) = \{q / \exists E_n \subseteq f'(e) \quad (n,q) \in f'(d)\}$$

$$= \{\underline{b} / \exists \beta \quad E_{g(\beta)} \subseteq f'(e) \qquad (g(\beta), \underline{b}) \in f'(d)\}$$

by (2), since $E_1 = \{0\}$ (so it is not the case that $(n,q) = \underline{a}$ for $a \in A$),

$$= \{\underline{b} / \exists \beta \subseteq e \qquad (\beta; b) \in d\}, \qquad \text{by (1)}$$

$$= f'(de) \quad \square$$

In Eng.[1979] it is shown that for any applicative structure $\langle A,x \rangle$, one has

$$\langle A,x \rangle \Rightarrow \langle D_A, \cdot \rangle$$

4.3 **Corollary.** Let $A \neq \emptyset$. Then for any countable $\langle E,x \rangle$, one has

$$\langle E,x \rangle \Rightarrow \langle D_A, \cdot \rangle$$

Proof. Just use $\langle E,x \rangle \Rightarrow \langle D_E, \cdot \rangle \Rightarrow \langle P\omega, \cdot \rangle \Rightarrow \langle D_A, \cdot \rangle$. $\square$

In particular $\langle E,x \rangle$ may be a countable extensional Combinatory Algebra.

4.4 **Discussion.** $P\omega$ is not (as) well founded (as D_A; see 2.5). Namely, there is no way to mimic the definition of $| \quad |$ given for D_A (see 2.4-5) on $P\omega$. If, in view of (0) in 4.2, one sets $|n| = k$ for $n \neq 0$, then $| \quad |$ cannot be extended to 0 in a way to have always $|n|,|m| < |(n,m)|$. In fact $0 = (0,0) = (0,(0,0))$....(and this is the only "bad guy", but any coding must have at least one...). Thus under the standard coding (but this can be generalized).

$$\forall C \in P\omega \quad \{0\}C = \{0\}$$

This is what we have been taking care of in 4.2(i). Clearly non well founded codings in even a stronger sense would make the result false (see BB[1979] for strongly non well founded codings). Note that the proof of 4.2 does not depend on properties of the "standard" codings, other than their almost well foundedness (this discussion continues in A.6, where a change in the definition of the set D_A, say D_A^0 gives $\langle P\omega, \cdot\rangle \simeq \langle D_A^0, \cdot\rangle$).

Theor. 2.7 and Hyland's result (Bar[1981], 19.1.19) show that $\langle P\omega, \cdot\rangle$ and $\langle D_A, \cdot\rangle$ can be turned into equationally equivalent λ-models. 4.2 tells us about isomorphic embeddings. Nonetheless in no case $\langle P\omega, \cdot\rangle$ and $\langle D_A, \cdot\rangle$ can be made isomorphic.

4.5 **Theorem** $\forall A \quad \langle P\omega, \cdot\rangle \not\simeq \langle D_A, \cdot\rangle$.

Proof. We first need a few remarks.

$P\omega$ Claims (1) $\quad \forall C \in P\omega \ \emptyset C = \emptyset$ and $\{0\}C = \{0\}$

(2) $\quad \forall E_n \neq \emptyset \quad \exists \vec{C} \quad E_n\vec{C} = \{0\}$

(3) $\quad \forall C \in P\omega \quad (C \text{ infinite} \Rightarrow \forall h \exists k > h \ \exists G_1,\ldots,G_k \quad C\vec{G} \neq \emptyset)$

(1) is obvious. As for (2), take the largest k such that $m_k \neq 0 \wedge (m_1,(m_2,\ldots(m_k,0)..) \in E_n$. Then $E_n E_{m_1} \ldots E_{m_k} = \{0\}$. (3) is trivial.

D_A Claims.

(4) $\quad \forall d \in D_A \ (\exists e \ de = d \Rightarrow d = \emptyset \vee d \text{ infinite})$

(5) $\quad \forall d \in D_A \ (\exists e \ de \text{ infinite} \Rightarrow d \text{ infinite})$

(6) $\quad \forall \beta \in D_A$, finite, $\exists h \forall k > h \ \forall d_1,\ldots,d_k \ \beta\vec{d} = \emptyset$

To prove (4), assume that $d \neq \emptyset$ is finite and take the largest n such that $(\beta_1;\ldots(\beta_n; a)\ldots) = b \in d$ for some $\vec{\beta}$,a. Then $\forall e \ b \notin de$. (5) and (6) are proved similarly.

Assume now that $f:P\omega\to D_A$ is an isomorphism, for some $A\neq\emptyset$. Let $\underline{K},\underline{S},\underline{I}$ interpret K,S,I in $P\omega$. Clearly $\langle D_A, \cdot, f(\underline{K}), f(\underline{S})\rangle$ is a Comb. Alg., namely it is a particular expansion of $\langle D_A, \cdot\rangle$, with interpretation, say, $[\![\ ]\!]^f_\sigma : CL \to D_A$. $\langle D_A, \cdot\rangle$ has approximable application (see 2.6, part 1), and this depends only on the properties of $\langle D_A, \cdot\rangle$ as applicative structure. Thus 1.9 applies and

$$\begin{aligned}
\emptyset &= [\![SII(SII)]\!]^f && \text{,by 1.9}\\
&= f(\underline{SII(SII)}) && \text{,by def. of } [\![\ ]\!]^f \text{ and } f\\
&= f([\![SII(SII)]\!]) &&\\
&= f(\emptyset) && \text{,by } P\omega\models SII(SII)=\emptyset \text{ (Bar[1981] ch.19.1)}
\end{aligned}$$

By assumption f is injective, thus, by (1) and (4), $f(\{0\})$ is infinite.

Finally, observe that

$$\forall C\neq\emptyset \qquad f(C) \text{ is infinite}$$

In fact: if C is finite, then use (2), the fact that $f(\{0\})$ is infinite and (5). If C is infinite, then use (3) and (6). This concludes the proof. □

I. Appendix A. Non Well Foundedness, Terms of Order 0, Isomorphisms with $\langle P\omega, \cdot \rangle$

The notation is as at the end of the Intermezzo, where $\tilde{D}_A = \langle \tilde{D}_A, \cdot \rangle$ was defined. In particular recall that, in $\tilde{D}_A$, $a = (\{a\};a)$. $\underline{\lambda}$ is as in 2.1(iii), i.e. $\underline{\lambda}x.f(x) = \{(\beta;b)/b \in f(\beta)\}$.

A.1 Lemma In $\tilde{D}_A$ one has:

(i) $Ad = A \cap d$.

(ii) $a \in d \wedge a \in e \Rightarrow a \in de$.

(iii) $d \subseteq \underline{\lambda}x.dx$

(iv) $\langle \tilde{D}_A, \cdot, \underline{\lambda} \rangle$ is a λ-model.

Proof. By the definitions. □

Given $a \in A$, let c_a be the constant symbol for $\{a\}$ and $\Lambda(c_a)$ the set of λ-terms built up using also c_a, where $[\![c_a]\!] = \{a\}$.

A.2 Lemma Let σ:var $\rightarrow \tilde{D}_A$ be constantly equal to $A \in \tilde{D}_A$. Then $\forall a \in A$ $\forall M \in \Lambda(c_a)$ $a \in [\![M]\!]\sigma$.

Proof (By induction). If $M \equiv c_a$ or $M \equiv x$, we are done. If $M \equiv PQ$, use A.1(ii) and the induction. If $M \equiv \lambda x.N$, then $[\![M]\!]\sigma = \{(\beta;b)/b \in [\![N]\!]\sigma_x{}^{\beta}\}$.
Notice now that $N[c_a/x] \in \Lambda(c_a)$; then
$a \in [\![N]\!]\sigma_x{}^{\{a\}} = [\![N[c_a/x]]\!]\sigma$, by induction, i.e.,
$a = (\{a\},a) \in [\![M]\!]\sigma$. □

A.3 Lemma Let σ be as in A.2. Then one has for any $M \in \Lambda$.

(i) $A \subseteq [\![M]\!]\sigma$

(ii) $A = [\![M]\!]\sigma \Rightarrow M \in 0_0$.

Proof.

(i) If $M \in \Lambda$, then $\forall a\ M \in \Lambda(c_a)$.

(ii) If $\lambda\beta \vdash M = \lambda x.N$, for some N, then $(\gamma;b) \in [\![M]\!]\sigma \wedge (\gamma;b) \subseteq (\beta;b) \Rightarrow (\beta;b) \in [\![M]\!]\sigma$. □

A.4 **Theorem**. Let σ be as in A.2. Then

$$\forall M \in 0_0 \qquad [\![M]\!]\sigma = A$$

Proof. Define first a labelled λ-calculus ΛI leaving out label 0: $M \in \Lambda I \Rightarrow M^n \in \Lambda I$, for $n \geq 1$.

Add the following rules:

I.1 $(M^p)^q \rightarrow M^{\min[p,q]}$
I.2 $(\lambda x.M)^{n+2} \rightarrow (M[N^{n+1}/x])^{n+1}$

If $b \in \tilde{B} = B/\simeq$, then b, as equivalence class in B, contains a shortest element (in B), say sh(b): this element is obtained by collapsing all $(\{a\},a)$ to a. Let $|\ \ |: B \rightarrow \omega$ be as in 2.5. Define, for $b \in \tilde{B}$ and $d \in \tilde{D}_A$, $|b|\uparrow = |sh(b)|$ and $d_n = \{b \in d / |b|\uparrow \leq n\}$.

Then 1,2,3,,5 of def. 1.3 hold. 4 holds for $n \geq 1$ (notice that $d_1 e \subseteq (de_1)_1 \subseteq (de)_1$ and that $d_1 \subseteq A$).

Interpret $M^n \in \Lambda I$, by $[\![M^n]\!]_\sigma = ([\![M]\!]_\sigma)_n$. Let I be a complete labeling of M iff M^I is M with a label on any subterm. Then, by usual arguments, one has:

(1) $M \underset{(I)}{\rightarrow} N \quad \Rightarrow \quad \forall\sigma\ [\![M]\!]_\sigma \subseteq [\![N]\!]_\sigma$

(2) $[\![M]\!]_\sigma = \cup\{[\![M^I]\!]_\sigma / I$ compl. lab. of $M\}$

Assume now that $M \in 0_0$.

Case $M \notin P0_0$. Then $\lambda\beta \vdash M = x\vec{N}$ and $A \subseteq [\![M]\!]_\sigma$, by A.3

$\subseteq A$, by A.1(i).

Case $M \in P0_0$. Let M^I be as above.

Then one has: (3) $M^I \rightarrow N^1\vec{Q}$, for some $N, \vec{Q}$.

Thus $A \subseteq [\![M]\!]_\sigma$,by A.3(1)
$\subseteq A$,by (2),(3) and (1)

□

A.5 Corollary Let $M \in \Lambda^0$. Then

(i) $\tilde{D}_A \models A \subseteq M$

(ii) $\tilde{D}_A \models A = M \Leftrightarrow M \in 0_0$.

Note that $Y_T(\underline{\lambda}x.x) = \emptyset$, thus Tarski's fixed point operator Y_T is not λ-definable.

A.6 **Discussion** $P\omega$ partially satisfyies 1-5 of 1.3, in the same way as $\tilde{D}_A$ does (see Bar.[1981], p.473 setting 1 for 0 and n+1 for n). Moreover 0=(0,0).... Why is it that, for $M \in 0_0{}^0$, $P\omega \not\models M = \{0\}$, but $P\omega \models M = \emptyset$? The point is that (0,0) corresponds to "($\emptyset$; 0)", not to "({0},0).
As a matter of fact, take $A=\{a\}$ and set $a \simeq (\emptyset;a)$. Then, for $B' = B/\simeq$ and $D_A{}^0 = 2^{B'}$, $\langle P\omega, \cdot\rangle \simeq \langle D_A{}^0, \cdot\rangle$. The isomorphism follows by the proof of $\langle P\omega, \cdot\rangle \Rightarrow \langle D_A, \cdot\rangle$ given in 4.3. Moreover set $\varepsilon = [\![\lambda xy.xy]\!] \in P\omega$ and $\varepsilon_A = \underline{\lambda}xy.xy = \{(\beta;(\gamma;b))/b \in \beta\gamma\} \in D_A^0$ then that isomorphism takes ε to ε_A. Using Meyer's algebraic approach, Mey[1981], this shows that $P\omega$ and D_A^0, with the ordinary application and abstraction, are also isomorphic as λ-models.

II. Appendix B. $D_A \models M \subseteq N \Rightarrow BT(M) \subseteq BT(N)$

This appendix completes the proof of 2.7, thus the notation is as in Chapter. 2.

B.1 <u>Lemma</u> (i) Let $f \in (D_A \rightarrow D_A)$. Then $(\beta;b) \in \underline{\lambda}x.f(x) \wedge (\beta;b) \subseteq (\gamma;b) \Rightarrow (\gamma;b) \in \underline{\lambda}x.f(x)$ (let's say that $\underline{\lambda}x.f(x)$ is <u>saturated</u>).

(ii) Let $\underline{A} = B \backslash A$. Then $(d \cap \underline{A})e = de$

(iii) $d \subseteq \underline{A} \Leftrightarrow d \subseteq \underline{\lambda}x.dx$.

<u>Proof</u> (i) By monotonicity of f. (ii) By definition. (iii) $\Rightarrow$. $(\beta;b) \in d \Rightarrow b \in d\beta$

$\Leftarrow$ $\underline{\lambda}x.dx$ does not contain elements of A. □

Note that (ii) and (iii) hold just because one can distinguish between elements of A and elements of $\underline{A}$. Fix now $a_0 \in A \neq \emptyset$.

B.2 <u>Def.</u> Let $f \in (D_A \rightarrow D_A)$. Define

$$\lambda^0 x.f(x) = \underline{\lambda}x.f(x) \cup \{a_0\}.$$

(<u>Notation</u>. For $f \in C(D_A^n, D_A)$, set

$$\lambda^0 x_1 \ldots x_n.f(x_1,\ldots,x_n) = \lambda^0 x_1(\lambda^0 x_2 \ldots x_n.f(x_1,\ldots x_n));$$

by the continuity of $\cup$ and $\underline{\lambda}$, this is a good definition).

B.3 <u>Remark</u> By definition

$$\underline{\lambda}x_1 \ldots x_n.f(x_1,\ldots,x_n) = \{(\beta_1;\ldots(\beta_n;b)\ldots)/b \in f(\beta_1,\ldots,\beta_n)\},$$

while $\lambda^0 x_1 \ldots x_n.f(x_1,\ldots,x_n)$ contains also a_0 and all elements of the type $(\beta_1;a_0),\ldots(\beta_1;\ldots(\beta_{n-1};a_0)$ for arbitrary β's.

B.4 <u>Lemma</u> Let $f \in C(D_A^n, D_A)$. Then

1) $0 \leq i \leq n \Rightarrow (\lambda^0 x_1 \ldots x_n.f(x_1,\ldots,x_n))d_1 \ldots d_i = \lambda^0 x_{i+1} \ldots x_n.f(d_1,\ldots d_i,x_{i+1}\ldots,x_n)$

2) $0 \leq p < n \Rightarrow a_0 \in (\lambda^0 x_1 \ldots x_n.f(x_1,\ldots,x_n))d_1 \ldots d_p$.

Proof. Easy. □

Notation Given $P \subseteq D_A^n$, $\beta_1,\ldots,\beta_n$ are minimal such (mims) $P(\beta_1,\ldots,\beta_n)$ iff

i) $P(\beta_1,\ldots,\beta_n)$ holds, and

ii) if $\gamma_1 \subseteq \beta_1,\ldots,\gamma_n \subseteq \beta_n$ and $\vec{\gamma} \neq \vec{\beta}$,
then $\neg P(\gamma_1,\ldots,\gamma_n)$.

B.5 **Def.** Let $f \in C(D_A^n, D_A)$. Define

$$\lambda^- x_1 \ldots x_n.f(x_1,\ldots,x_n) = \{b \in \lambda^0 x_1 \ldots x_n.f(x_1,\ldots,x_n) / \\ / \exists c\, \exists \vec{\beta}\ b = (\beta_1;\ldots(\beta_n;c)\ldots) \Rightarrow \vec{\beta} \text{ mins } c \in f(\vec{\beta})\}$$

B.6 **Lemma** Let $f \in C(D_A^n, D_A)$. Then

1) $(\lambda^- x_1 \ldots x_n.f(x_1,\ldots,x_n))d_1 \ldots d_n = f(d_1,\ldots,d_n)$

2) If $\forall d_1,\ldots,d_{n-1}\ \exists d_n\ f(d_1,\ldots,d_{n-1},d_n) \neq \emptyset$, then

$(0 \leq p < n \Rightarrow (\lambda^- x_1 \ldots x_n.f(x_1,\ldots,x_n))d_1 \ldots d_p$ contains a_0 and it is not saturated (cf. B.1)).

Proof. 1) By definition and B.4(1).

2) Let's write $F_n^- := \lambda^- x_1 \ldots x_n.f(x_1,\ldots,x_n)$. Then

$$a_0 \in F_n^- d_1 \ldots d_p = \{c / \exists \vec{\beta} \subseteq \vec{d}\ (\beta_1;\ldots(\beta_p;c)\ldots) \in F_n^-\},$$

since $p < n$ and by B.4(2) and the definition of F_n^-.

As for "non saturation", notice first that

$$f(e_1,\ldots,e_n) = F_n^- e_1 \ldots e_n, \qquad \text{by (1)}$$

$$= \{b / \exists \vec{\beta} \subseteq \vec{e}\ (\beta_1;\ldots(\beta_n;b)\ldots) \in F_n^-\}; \text{ and}$$

hence, by the assumption on f,

$$\forall p<n \forall e_1,\ldots,e_p\ \exists\beta_1\subseteq e_1 \ldots \exists\beta_p\subseteq e_p\ \exists\beta_{p+1}\ldots\ \exists\beta_n\ \exists b\ (\beta_1;\ldots(\beta_n;b)\ldots)\in F_n^-$$

Recall now that by definition of F_n^-, these $\beta,\ldots,\beta_n$ are "minimal such"; thus, in particular, $\forall\gamma\supset\beta_{p+1}\ (\gamma;(\beta_{p+2};\ldots(\beta_n;b)\ldots)\notin F_n^- d_1\ldots d_p$. □

B.7 __Def.__ Let $C_p^- \equiv \lambda^- x_0\ldots x_{p+1}.x_{p+1}x_0\ldots x_p$.

Let $\Lambda(D_A)$ the set of λ-terms built up using also constants (symbols) from D_A.

B.8 __Proposition__ (i) $D_A \models C_p^- x_0\ldots x_{p+1} = x_{p+1}x_0\ldots x_p$

(ii) Let $\sigma_1,\ldots,\sigma_m$ and $\tau_1,\ldots,\tau_q$ be terms in $\Lambda(D_A)$. If $n\neq t$ and $m,q<p$, then

$$D_A \not\models \lambda x_1\ldots x_n.C_p^-\sigma_1\ldots\sigma_m \subseteq \lambda x_1\ldots x_t.C_p^-\tau_1\ldots\tau_q.$$

__Proof.__ (i) by B.6(1).

(ii) Clearly $f(d_0,\ldots,d_{p+1}) = d_{p+1}\ d_0\ldots d_p$ satisfies the conditions on f in B.6(2)

Let $n<t$.

__Case 1__ Assume $.\supseteq.$ and apply both LHS and RHS to $x_1,\ldots,x_n$. Then

$$D_A \models C_p^-\sigma_1\ldots\sigma_m \supseteq \lambda x_{n+1}\ldots x_t.C_p^-\tau_1\ldots\tau_q.$$

This is impossible since the LHS is not saturated by B.6(2), while the RHS is saturated by B.1(i).

__Case 2__ Assume $.\subseteq.$ As above one obtains

$$D_A \models C_p^-\sigma_1\ldots\sigma_m \subseteq \lambda x_{n+1}\ldots x_t.C_p^-\tau_1\ldots\tau_q,$$

which is impossible since the LHS contains a_0 by B.6(2), while the RHS doesn't. □

B.8 is the C-lemma, 3.3, of BL[1980].

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