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I. INTRODUCTION

A family $C_1,\ldots,C_m$ of simple circuits of an undirected multigraph $G = (V,E)$ *covers* G provided each edge of G is in one of the circuits. The *size* of such a family is then the sum of the lengths of the circuits $C_1,\ldots,C_m$. We are interested here in the question of finding covers of minimum size. Clearly, we can restrict our attention to 2-connected multigraphs: if a graph has a bridge then it has no cover at all.

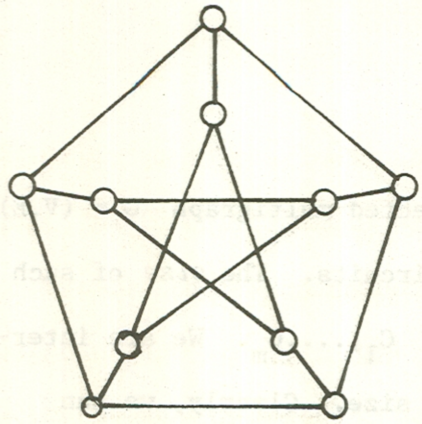
This problem bears a superficial similarity to the Chinese postman problem, in which one seeks to find the minimum number of edges that have to be added to G so as to result in an Eulerian multigraph. The difference is best exhibited by the famous Petersen graph (fig. 1a). There is an Eulerian supergraph of this graph with 20 edges, and this is best possible (fig. 1b). However, its minimum size cover is 21 (fig. 1c).

This problem of minimum cover size was first considered in [IR] where its application to irrigation systems was described. It was shown in [IR] that there is always a cover size $|E| + 2|V| \log |V|$ and that this cover can be found in average time $O(|V|^2)$. Here we will show that every 2-connected multigraph has a cover of size

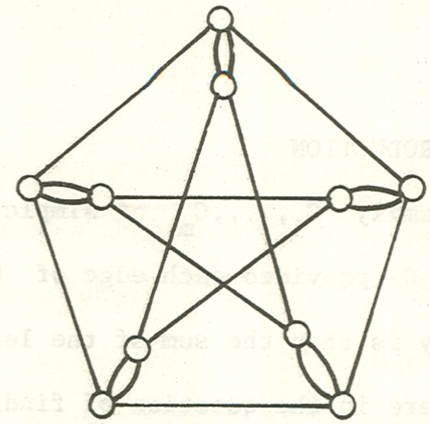
$$\min \{3|E| - 6, \quad |E| + 6\cdot|V| - 7\}$$

thus improving the previous results for sparse graphs. This cover can be found in $O(|V|^2)$ time.

Our construction relies heavily on that used by Jaeger [Ja] for showing that every 2-edge connected graph has a nowhere-zero flow modulo 8. Jaeger's paper does not contain full proofs and algorithms and hence, many of our results are only motivated by his. Matthews [M] also deals with a related problem, however he misquotes Jaeger and is also subsumed by Jaeger.



a



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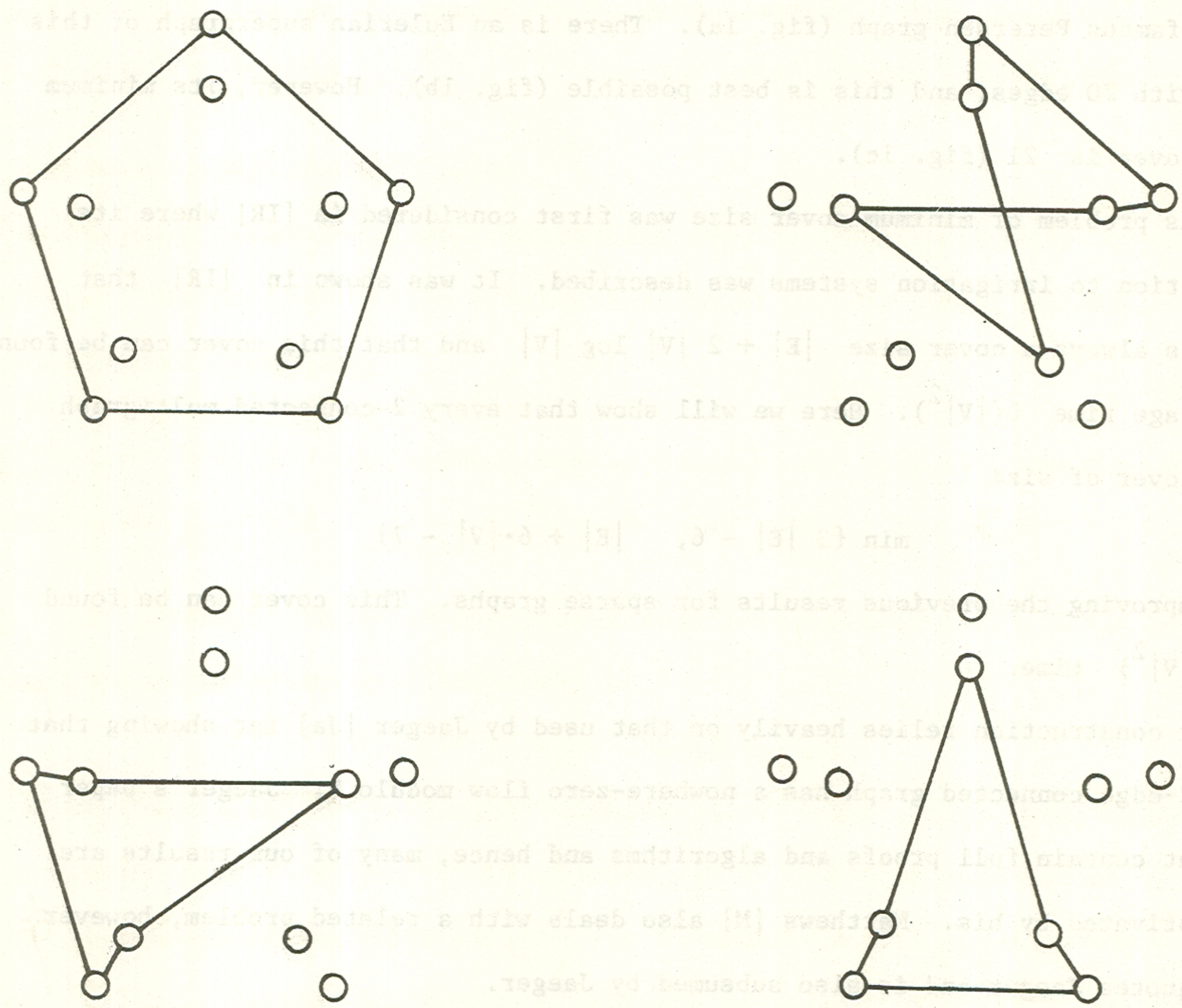


c

In Section 2 we show that in order to find a small size cover for a dense multigraph it suffices to find one for an efficiently extracted sparse one. We also give a general technique, whereby, given a spanning tree T, one can find a cover of all the edges except perhaps for certain edges of T. In the next section we show that if the multigraph is 3-edge connected, then its edges can be covered by three Eulerian subgraphs. Then in the next-to-last section we extend this result to 2-edge connected multigraphs, which yields the bound sought. Finally we show that our cover can be found in $O(|V|^2)$ time.

2. REDUCTION TO SPARSE GRAPHS

If a graph is sparse (i.e., $|E| = O(|V|)$) then it seems reasonable to expect that there exists a cover of size $O(|E|)$. Therefore, following [IR], we will reduce the general problem to that of sparse multigraphs. As usual, an *Eulerian subgraph* of G is a subgraph of G consisting of edge-disjoint circuits (notice it need not be connected).

Lemma 1: Let $T = (V,E_T)$ be a spanning tree of $G = (V,E)$. Then there exists an Eulerian subgraph $H_0 = (V,E_{H_0})$ of G with $E_{H_0} \supseteq E - E_T$.

Proof: E_{H_0} is constructed by successively deleting edges. Initially $E_{H_0} = E$. We then perform a depth first search (DFS) on T. Each tree edge is traversed twice: once in the forward direction and once backwards. If when traversing a tree edge (u,v) backwards from v to u the degree of v is odd in the current E_{H_0}, then delete the edge (u,v) from E_{H_0}. On termination, all vertices are of even degree in E_{H_0}, and hence, $H_0 = (V,E_{H_0})$ is an Eulerian subgraph of G. □

Note that the construction of this lemma can be done in $O(|E|)$ time whether T is a DFS tree or not.

In order to find a cover of G, one has to cover the edges of $E - E_{H_0} \subseteq E_T$. However, T is not a 2-edge connected multigraph and our method is not immediately applicable to it. We therefore augment T into such a graph.

Suppose that T is a DFS tree. We call an edge (u,v) of $E - E_T$ a *lowest frond* if u is the ancestor of all vertices w for which $(v,w) \in E - E_T$. Let us define the graph $H = (V, E_H)$ where E_H is T and all the lowest fronds. H is then 2-connected, has at most $2|V| - 2$ edges, and contains all the uncovered edges of H_0 as required. We summarize this as follows:

Corollary 1: Suppose that one can find in time $t(|V|,|E|)$ a cover of size $s(|V|,|E|) + |E|)$ for any 2-connected multigraph $G=(V,E)$. Then we can find a cover of size $|E| + s(|V|, 2|V|$ in time $t(|V|, 2|V| - 2) + O(|E|)$.

3. COVERING A 3-EDGE CONNECTED GRAPH

The following lemma could have followed directly by applying Edmonds' Matroid Partitioning Theorem [Ed1] to the co-tree matroid of G. Our proof, however, suggests a more efficient algorithm.

Lemma 2: Let $G=(V,E)$ be a 3-edge connected multigraph. Then G contains three spanning trees $T_i = (V, E_{T_i})$ $(i = 1,2,3)$ such that $E_{T_1} \cap E_{T_2} \cap E_{T_3} = \emptyset$.

Proof: Let $D = (V,A)$ be the directed multigraph derived from G by replacing each edge by two arcs, one for each direction:

$$A = \{(u,v) \mid (u,v) \text{ is edge of } G\}.$$

Let v be a vertex of V. By Menger's Theorem on G there exists three edge-disjoint paths from v to any triple of nodes of V. Hence by the Theorem of Edmonds [Ed2] there exists three arc-disjoint directed trees B_1, B_2, B_3 rooted at v. Let T_1, T_2, T_3 be the underlying undirected trees. Each edge of G corresponds to two arcs of D, and hence it can appear in at most two of the undirected trees. Thus, $T_1 \cap T_2 \cap T_3 = \emptyset$ □

Using Lemma 1, each of these trees $T_i = (V, E_{T_i})$ of Lemma 2 induces a cover C_i of $(V, E - E_{T_i})$. Therefore, $C_1 \cup C_2 \cup C_3$ is a cover of G. □

Theorem: Let G be a 3-edge connected multigraph. Then G can be covered by three Eulerian subgraphs.

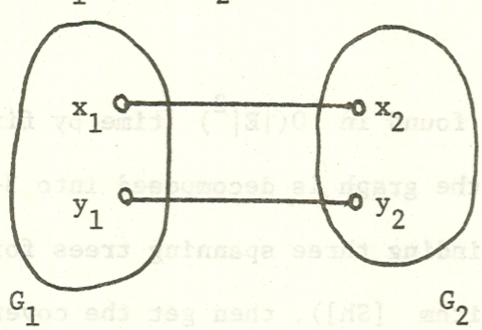
Using Tarjan's algorithm [Ta] the tree can be found in $O(|V| \cdot |E|)$ time. For dense graphs Shiloach's algorithm [Sh] finds the trees faster (in time $O(|V|^2)$), but this is immaterial here since by Corollary 1 we need consider only sparse graphs. (i.e., $|E| = O(|V|)$).

4. COVERING A 2-EDGE CONNECTED GRAPH

Theorem 1 can be strengthened to yield the following:

Theorem 2: Let $G = (V,E)$ be a 2-edge connected multigraph. Then G can be covered by three Eulerian subgraphs.

Proof: We proceed by induction on the number of nodes of G. If $|V| = 1$ then the result is obvious since G consists of loops only. Now consider a multigraph $G = (V,E)$ with $|V| > 1$. By Theorem 1 we can assume that G is not 3-edge connected. Thus there exists a pair (x_1, x_2) and (y_1, y_2) of edges that disconnect G into two components G_1 and G_2 (fig.2).



Now create two new multigraphs G_1' and G_2' by deleting these edges and replacing them by two new edges (x_1, y_1) and (x_2, y_2). Since G_1' and G_2' both have fewer than $|V|$ vertices, by our induction hypothesis they can be covered by three Eulerian subgraphs each. Let these Eulerian subgraphs have edge sets E_1, E_2, E_3 (for G_1') and F_1, F_2, F_3 (for G_2'). By renaming we may assume that

$$(x_1,y_1) \in E_1,\ldots,E_i$$

$$(x_1,y_1) \notin E_j \qquad j > i$$

$$(x_2,y_2) \in F_1,\ldots,F_{i+k}$$

$$(x_2,y_2) \notin F_j \qquad j > i+k.$$

Without loss of generality assume $k \geq 0$. There are two cases:

Case 1: $(k=0)$. Then the following S_1, S_2, S_3 is a cover for G:

$$S_j = \begin{cases} E_j \cup F_j - \{(x_1,y_1),(x_2,y_2)\} \cup \{(x_1,x_2),(y_1,y_2)\} & \text{if } (x_1,y_1) \in E_j \\ E_j \cup F_j & \text{otherwise} \end{cases}$$

Case 2: $(k>0)$. Then we find a new cover F_1', F_2', F_3' of G_2' by replacing $F_{i+1},\ldots,F_{i+k}$ by

$$F_i \oplus F_{i+1},\ldots,F_i \oplus F_{i+k}$$

Then the new covers E_1, E_2, E_3 and F_1', F_2', F_3' satisfy the conditions of case 1. □

5. TIME BOUNDS

The cover of Theorem 2 can be found in $O(|E|^2)$ time by first finding separating pairs of bridges repeatedly until the graph is decomposed into 3-edge connected graphs (time $O(|V|\cdot|E|)$, then finding three spanning trees for each component (time $O(|V|^2)$ by Shiloach's algorithm [Sh]), then get the cover by Lemma 2 and finally combine the partial solutions together. This last may possibly require rearranging the Eulerian subgraphs as in case (b) of Theorem 2 (time $O(|E|)$). If we first use the reduction to sparse graphs by Corollary 1, then the entire algorithm runs in time $O(|V|^2)$.

6. CONCLUSIONS

We show that any 2-edge connected multigraph can be covered by three Eulerian subgraphs. If three Eulerian circuits are required, then each may contain at most $|E| - 2$ edges; therefore, any graph with $|E|$ edges can be covered with a set of circuits of total length $3|E| - 6$.

If we apply the reduction of Section 2, then the graph $H_0 = (V, E_{H_0})$ has at most $|E| - 1$ edges (otherwise we are done); whereas $H = (V, E_H)$ has at most $2|V| - 2$. Therefore the total number of edges in the three Eulerian subgraphs of H and H_0 is $|E| + 6|V| - 7$.

Several problems remain open. There is no know graph which requires covers of size significantly larger than E. Thus, one may expect that the multiplicative constants in our bound

$$\min\{|E| + 6 \cdot |V| - 7,\ 3|E| - 6\}$$

can be improved. It seems that the additive constants can be improved quite easily.

Finding the three spanning trees requires time $O(|V|^2)$, and this dominates the time bound. The reduction of Section 4 also requires $O(|V|^2)$ time. However, an alternative based on Jaeger's original construction and the partition of the graph into tri-connected components [HT] would require $O(|E|)$ time.

Finally, nothing is known about the complexity of minimizing the size of the cover of G.

7. REFERENCES

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