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CONSTRUCTION HEURISTICS FOR GEOMETRY AND A VECTOR ALGEBRA REPRESENTATION OF GEOMETRY

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ABSTRACT

Heuristics for generating constructions to help solve high school geometry problems are given. Many examples of the use of these heuristics are given. A method of translating geometry problems into vector algebra problems is discussed. The solution of these vector algebra geometry problems is analyzed. The use of algebraic constructions to help solve these vector problems is also discussed.

ACKNOWLEDGEMENT

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1. INTRODUCTION

In the field of mathematics there are some problems which cannot be solved using only the elements defined in the problem. When this situation occurs, it becomes necessary to introduce some additional elements into the problem. These additions to the problem are usually known as constructions.

In elementary algebra the elements of a problem are the variables in it. The introduction of new elements into the problem corresponds to the definition of some new variables in the problem. In the word problems of modern algebra the elements are the members of a group. The insertion of a term of the form ($a\ a^{-1}$) corresponds to the introduction of a new element into the problem. In Euclidean Geometry the introduction of a new element corresponds to the introduction of some new points and lines into the figure.

In this thesis we will study heuristics for the generation of two kinds of constructions. In the first half of this thesis, some heuristics for creating geometry constructions will be discussed. These heuristics were developed to help solve geometry problems which satisfy the following 2 conditions

1) All lines in the problem figure are straight. There are no curved lines of any sort.

2) The problem cannot be solved by making the trivial construction of connecting 2 points* in the diagram.

Three types of construction heuristics will be described. The first type deals with the reflection of the figure as a construction. Heuristics concerning when and how to reflect part of the figure around a point will be discussed.

*Unless otherwise specified in this paper, the word "point" is understood to mean a position marked by the intersection of 2 or more lines, not just a position in a line.

The second type of heuristic is the Situational Construction Heuristic (SCH). The SCH's describe specific local situations in geometry (such as the type of goal or the types of constraints in the problem) and suggest the constructions that should be drawn for each situation. The reflection heuristics can also be considered an SCH. Since a reflection is not normally considered a construction, it will be discussed separately.

The third type of heuristic concerns constructions for the application of a previously proven theorem. Occasionally in geometry, the student will recognize that the problem figure is very similar to a figure of a previously proven theorem. The student may consider the information contained in the theorem to be useful for the solution of his own problem. In order to utilize this theorem, constructions must be inserted in the problem figure to create the figure of the previously proven theorem. Methods for drawing this type of construction are discussed.

In the second half of this thesis, geometry problems expressed in a different form are studied. There is an interesting alternative method of representing geometry problems. Through a simple transformation procedure we can convert a problem represented in geometric terms into one represented in vector algebra terms. The problem is then changed from proving geometric relations in a given figure to deducing algebraic relations from a set of simultaneous vector algebra equations. So, essentially, the geometric constraints of the figure are converted into vector equations. In this algebraic system, the geometric construction corresponds to the introduction of a new variable and new equations. In this half of the thesis, the solution of geometry problems in this vector algebra system is studied. Also, the algebraic version of the construction is studied. The geometry construction heuristics previously described are converted to a vector algebra form. The form and use of these algebraic construction heuristics are described.

2 GEOMETRY CONSTRUCTION HEURISTICS

2.1 Midpoint Reflection Construction

This discussion of the midpoint reflection construction is divided into four main sections. The first part will be concerned with a formal working definition of reflection.* Then using this definition, some general properties of reflection will be discussed. The second part is a discussion of when and how to apply the reflection constructions. Since the reflection can be applied to each point of the figure independently of the other points, some rules to determine which points to reflect will be given. Also some heuristics will be given to help determine which figures should have the construction applied to them. The third main part will be a discussion of the motivation for this construction. Reflection's method of operation will be analyzed. Examples of the use of the reflection construction on actual geometry problems will be given in part four.

2.1.1 A Formal Definition of Midpoint** Reflection

Although most people understand the concept of reflection, a formal definition of midpoint reflection will now be given so that a common definition can be used and referred to. This definition will also be used to prove some properties of the reflection. We can formally define the midpoint reflection construction in the following way.

> To reflect around a point M, find the images of all points X that are distinct from M. The following procedure should be followed

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*In this section I shall frequently refer to midpoint reflection as just reflection.

**Although this operation should really be called a point reflection, we shall continue to call it a midpoint reflection to emphasize the fact that reflection will only be applied to points that are midpoints. For a discussion of why reflection is applied only to midpoints see Section 2.1.4.

to find the image of a point X: draw the straight line determined by X and M, extend the straight line determined by X and M, for a length equal to the length XM. Now the endpoint of the extended line segment is X', the image of X reflected around M. The image of M, M', is the same point as M.

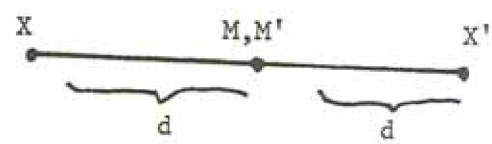


Fig. 1

Applying the midpoint reflection construction to a figure essentially creates an exact duplicate of the original figure. This duplicate is attached to the original at three points. These points are: M, the midpoint about which the reflection took place, and A and B, the 2 endpoints of the line segment of which M is the midpoint. Intuitively, we can regard the reflection as creating a duplicate of the original figure and them superimposing the duplicate so that the points M and M', A and B', and B and A' coincide.

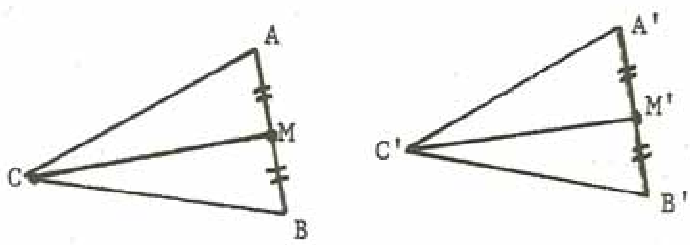


Fig. 2a

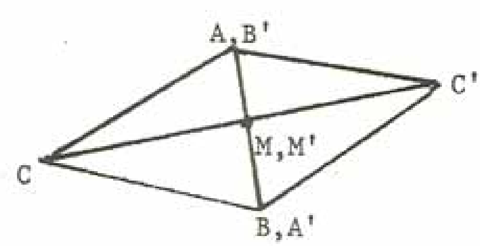


Fig. 2b

From our intuitive notion of the reflection in Fig. 2b we can observe some interesting properties of the reflection. For instance, in Fig. 2b, since AC'A'C is a parallelogram, so AC = A'C' and AC || A'C'. Using the formal definition of the midpoint reflection construction we can prove some of the relationships between elements of the figure and their images under reflection.

2.1.2 Midpoint Reflection Relations

<u>RR1</u> If XY is a line segment and X'Y' is the image of XY under midpoint reflection, the length of X'Y' is equal to the length of XY.

PROOF: If X, Y, and M are all colinear, XM = MX' and YM ' MY' by definition of the reflection construction.

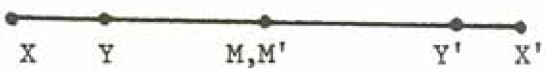


Fig. 3

So XY = (XM - YM) = MX' - MY') = X'Y'.

If X, Y, and M are not all colinear, connect line segments XM, YM, and XY. Then reflect the figure around M.

Fig. 4

Now, XM = MX' and YM = MY' by definition of the reflection construction. Angle XMY = angle Y'MX' by the equality of vertical angles. So triangle XMY is congruent to triangle X'MY' by SAS. Therefore, XY = X'Y' by corresponding parts of congruent triangles.

<u>RR2</u> If XY is a line segment and X'Y' is the image of XY under midpoint reflection, and if X, Y, and M are not all colinear, then XY is parallel to X'Y'.

PROOF: By the steps used to demonstrate RR1, we can also demonatrate that in Fig. 4 angle XYM = angle X'Y'M by corresponding parts of congruent triangles. So XY is parallel to X'Y' by the alternate interior angle theorem.

<u>RR3</u> If angle XYZ has as its image under midpoint reflection angle X'Y'Z', then angle XYZ = angle X'Y'Z'.

PROOF: If X, Y, and Z are not all colinear, connect line XZ so that angle XYZ is contained in triangle XYZ.

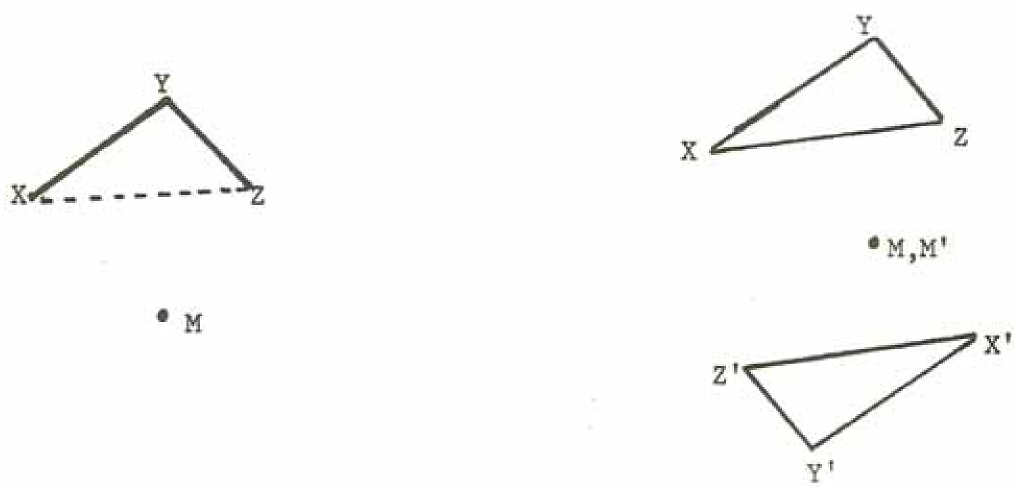


Fig. 5

As a result of the reflection, there will be a triangle X'Y'Z'. Now XY = X'Y', XZ = X'Z', and YZ = Y'Z' by RR1. So triangle XYZ is congruent to triangle X'Y'Z' by SSS. So by corresponding parts of congruent triangles, angle XYZ = angle X'Y'Z'. In order for this to be a valid argument, M, the point about which the reflection takes place, may be any point of the figure. It can even be the point X, Y, or Z.

PROOF: If X, Y, and Z are all colinear, find a 4th point W which is not colinear with X, Y, and Z.

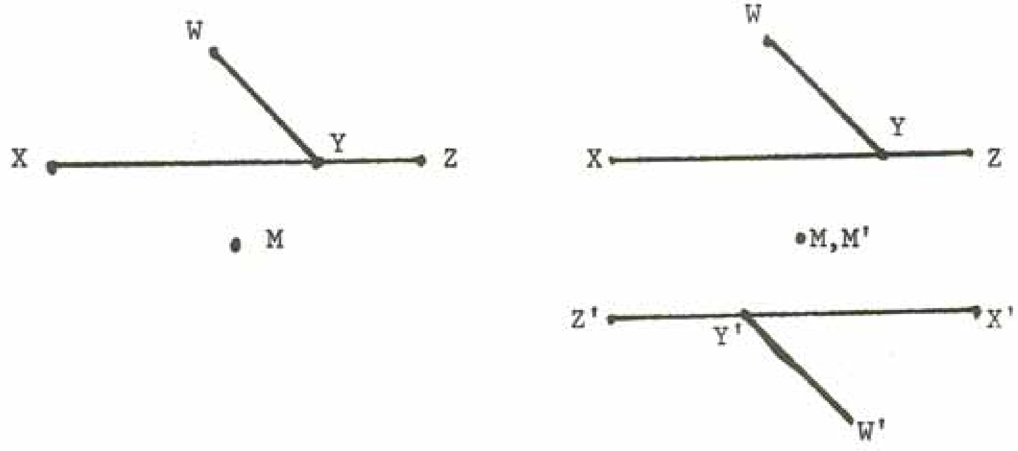


Fig. 6

Now since W is not colinear with X, Y, and Z, by the argument just completed above, angle XYW = angle X'Y'W' and angle WYZ = angle W'Y'Z'. So angle XYZ = angle XYW + angle WYZ = angle X'Y'W' + angle W'Y'Z' = angle X'Y'Z'.

2.1.3 When and How to Apply the Midpoint Reflection Construction

Now the problem of when and how to apply the midpoint reflection construction will be discussed.

The main criteria for applying the reflection construction is that there be a midpoint in the figure. A figure is a candidate for applying reflection whenever it contains a midpoint.

Once a midpoint to reflect around has been found, the next problem is to decide what parts of the figure to reflect. One simple solution is to reflect the entire figure. This method has the disadvantage of reflecting parts of the figure which may never be used in the proof. These unused figure parts do, however, increase the complexity of the diagram and make the proof harder to find.

We will now present an alternate method of performing the reflection. A set of rules will be given to determine which points in the figure to reflect. This method enables us to decide to reflect only the essential parts of the figure. So from the standpoint of complexity in the figure and in finding the proof, this method of reflection is superior to the simple method given above.

Embedded in the following set of rules are some heuristics to reject some figures as reflection candidates.

The following steps implement the rules to choose which points to reflect:

<u>ST1</u> Choose a midpoint in the figure. Call this point M. Let P and Q be the end points of the line which has M as its midpoint. Choose these 3 points.

<u>ST2</u> Select one of the other lines which intersect M. Find the 2 points

on this line that are the 2 closest points to M on either side of M. That is, one point is the closest to M from one side, the other point closest from the other side. Choose one of these 2 points. Call this chosen point E. In choosing a point E, preference should be given to points which are midpoints.

<u>ST3</u> After a point E has been chosen, check and see if E is the midpoint of some other line in the figure. If E is the midpoint of line segment GH, choose point G. Then go to step ST7. If E is <u>not</u> the midpoint of some line in the figure, this step does not apply.

<u>ST4</u> If the point E will be reflected into another point which is already in the figure, (i.e., if there is a point X and E' = X) then reject this point E and go to ST6.

If the point E will <u>not</u> be reflected onto another point in the figure, then this step does not apply.

<u>ST5</u> If the line EM is perpendicular to PQ (i.e., if EM is the perpendicular bisector of PQ), reject this point E and go to ST6. Otherwise this step does not apply and go to ST7.

<u>ST6</u> Go back to ST2 and select another point E. If another suitable point E cannot be found, then give up reflecting around the midpoint chosen in ST1. If there is another midpoint in the figure to reflect around, start at ST1 and try it. Otherwise give up on trying to apply the midpoint reflection construction to this figure.

<u>ST7</u> At this time, we have chosen the points P, Q, M, E and maybe G. All these chosen points should be reflected around M. Also, all

line segments formed by these chosen points should be reflected (formed implies a line segment which has 2 chosen points as its endpoints).

ST8 Now, usually the goal of the problem is to prove an equality or inequality between segments or that two segments are parallel. If the goal is this type, then look at all of the lines in the goal, if none of these segments are to be reflected, go back to ST1 and try to generate a new set of points and lines to reflect. If no set of lines to reflect can be generated which contains at least one of the segments in the goal, give up on trying midpoint reflection as a construction. If the goal concerns an angle equality or some other non-segment goal, this step does not apply.

After all these steps are finished, the reflection construction is complete.

2.1.4 Motivation for the Midpoint Reflection Construction

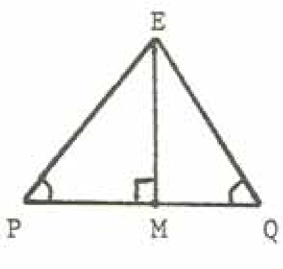
In order to motivate the reflection construction, it is first necessary to examine the effects of the construction. The construction reflects part of the figure around a midpoint. Through the midpoint reflection relations we can see that the image of what is reflected is identical to the original. Therefore, in a way, we can say that the reflection transports part of the figure to a different position. We could also say that the construction has reorganized the figure. Generally, the purpose of reflection and the reorganization of the figure is to regroup the elements of the goal and the elements constraints in such a way that they are all present in one single polygon (either a triangle or parallelogram). It is also intended that this grouping in a single polygon make the deduction of the goal be trivial.

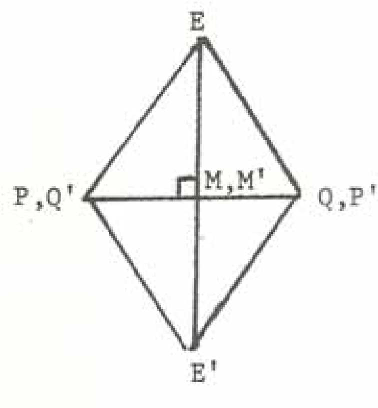
For example, if we wanted to prove 2 lines parallel, the intended result of a reflection construction would be a parallelogram with the 2 lines as opposite sides. If we wanted to prove two segments equal and were given 2 equal angles, the intended result of a reflection would be an isosceles triangle with the 2 equal angles as the base angles and the two segments as the legs of the isosceles triangle.

The decision of whether a particular figure can successfully be regrouped into the single polygon form is made when choosing which points to reflect. The rules to chose which points to reflect also contain heuristics to decide when a problem can be solved by reflection.

Step ST4 rejects an application of reflection because in the situation to which this step is applicable, all lines created by reflection can also be created by the trivial construction of connecting 2 points.

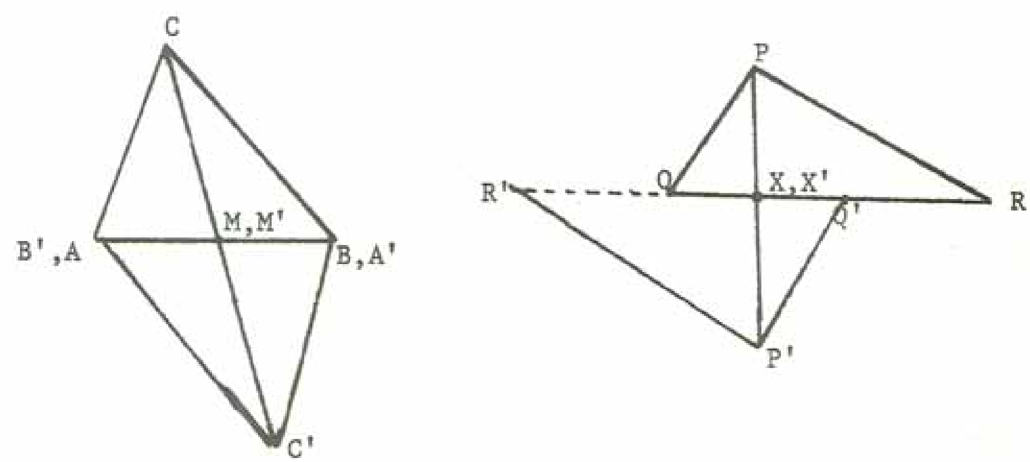
Step ST5 rejects an application of reflection because the reflection of a point E such that EM ⊥ PQ will not accomplish anything. The only relationships that could be deduced after the reflection would be those that could be deduced before the reflection. Essentially the regrouped figure created by reflection would be the same as the original figure.





Step ST8 rejects an application of reflection because in order that all the elements of the goal be present in a single polygon, it is usually necessary to regroup one of the elements of the goal.

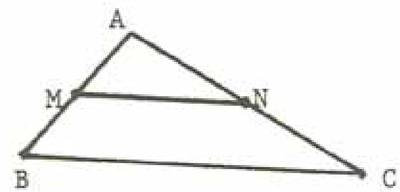
Reflection is applied only to points that are midpoints in the figure for the following reason. Reflection when applied to midpoints maps some points in the figure onto other points already in the figure. For example, if M is the midpoint of AB, then reflection around M will map A onto B. This mapping of points onto other points is crucial if we wish to achieve the effects described in the first paragraph of this section (the reorganization of the elements of the figure).



For example, suppose M is the midpoint of AB but X is not the midpoint of QR. The effect of the reflection around M is to regroup the figure so that segments AC and BC and angles ACM and BCM (or their equivalent images) are present is a single triangle, triangle BCC'. The reflection around X does not produce such a neat reorganization of the elements of the figure.

2.1.5 Examples of the Midpoint Reflection Construction

EXAMPLE 1



GIVEN: triangle ABC, AM = MB, AN = NC

PROVE: MN || BC and MN = $\frac{1}{2}$(BC)

In order to try reflection, use the rules to choose which points to reflect. Applying the rules we get:

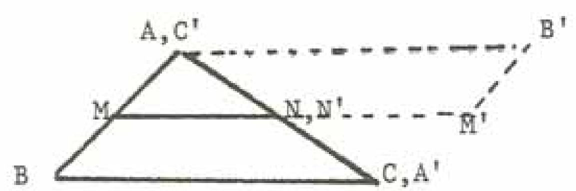
ST1 N is the midpoint of AC. Choose N, A, and C to reflect.

ST2 Line MN intersects N. Choose M to reflect.

ST3 M is the midpoint of AB. Choose B to reflect.

ST7 Points chosen: N, A, C, M, and B.

ST8 Since segment MN is part of the goal and is also to be reflected, we can proceed with the construction.



By RR1, BM = B'M'.

It is given that AM = BM.

Therefore B'M' = AM.

By RR2, AM is parallel to B'M'.

Therefore AB'M'M is a parallelogram.

So MM' = AB' and MM' is parallel to AB'.

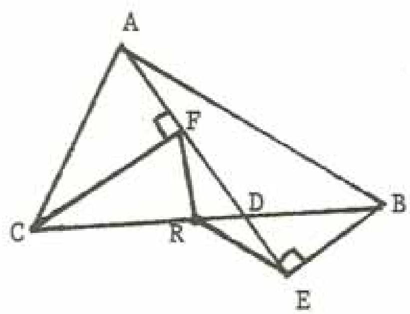
By RR2, AB' is parallel to A'B.

So MM' is parallel to A'B, or MN is parallel to BC.

By RR1, AB' = A'B and MN = M'N'.

So 2MN = A'B, or MN = $\frac{1}{2}$(BC).

<u>EXAMPLE 2</u>



GIVEN: triangle ABC, AD is any line drawn from A to the base BC, CF ⊥ AE, BE ⊥ AE, CR = RB

PROVE: RF = RE

Let us try reflection. We will apply the rules to choose reflection points.

ST1 R is the midpoint of BC. Choose R,C, B,.

ST2 RF intersects R. Choose F.

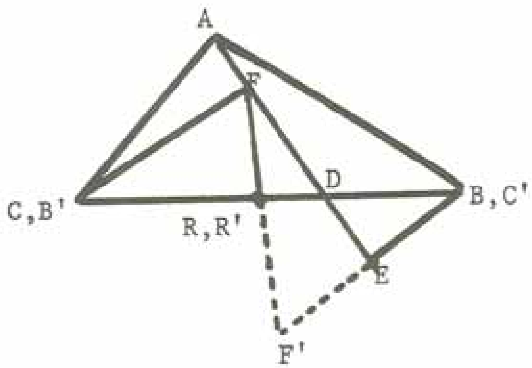
ST3 Does Not Apply (DNA).

ST4 DNA

ST5 DNA

ST7 Points chosen are R, C, B, F.

ST8 Since RF is part of the goal and is also to be reflected, the reflection can proceed.



By RR2, F'C' || FC || BE || C'E.

Now the postulate that only one line can be drawn parallel to a given line through a given point implies that F'C' and EC' must lie on the same straight line, so F'EC' is a straight line.

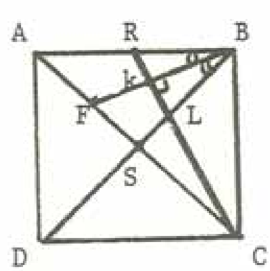
So FF'E is a right triangle.

By RR1, FR = F'R'.

Now use the theorem that the median to hypothenuse of a right triangle is equal to one-half the hypothenuse.

So FR = RE.

EXAMPLE 3



GIVEN: ABCD is a square, BF bisects angle DBA, CK ⊥ BF

PROVE: AR = 2SL

Let us try reflection.

ST1 k is the midpoint of RL. Choose k, R, L.

ST2 BF intersects k. Choose B.

ST3 DNA

ST4 DNA

ST5 Bk is perpendicular to RL so we reject point B.

ST6 We will go back to ST2 and choose another point.

ST2 BF intersects k. Choose F.

ST3 DNA

ST4 DNA

ST5 Fk is perpendicular to RL so we reject point F.

ST6 We have run out of points to choose in ST2. But all is not lost since there is another to reflect around. We will go back to ST1.

ST1 S is the midpoint of BD.

ST2 AC intersects S. Choose A.

ST3 DNA

ST4 A will be reflected onto C so we will reject A.

ST6 We can go back to ST2. The other points we can choose are F and C. Both of these will also be rejected. For the sake of brevity, we will not list those steps. After these rejections We can go back to ST1.

ST1 S is the midpoint of AC. Choose S, A, C.

ST2 LS intersects S. Choose L.

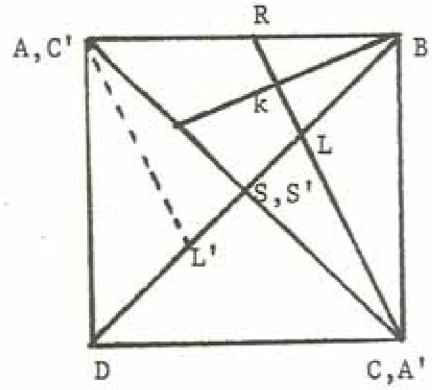
ST3 DNA

ST4 DNA

ST5 DNA

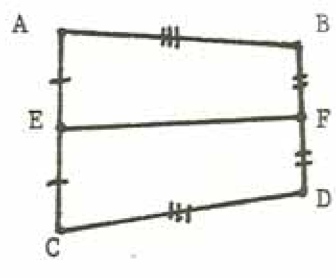
ST7 Points chosen S, A, C, L.

ST8 Since SL is part of the goal and is to be reflected, the reflection can proceed.



Now the problem can be solved.

EXAMPLE 4



GIVEN: AB = CD, AB is not parallel to CD, AE = EC, BF = FD

PROVE: the angle AB makes with EF is equal to the angle CD makes with EF

Let's try reflection.

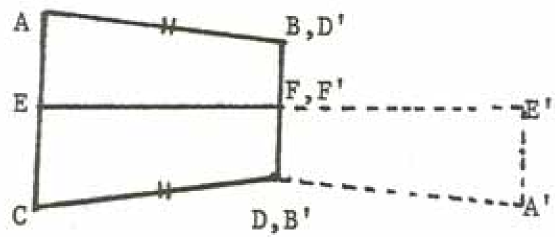
ST1 F is the midpoint of BD. Choose F, B, D

ST2 EF intersects F. Choose E.

ST3 E is the midpoint of AC. Choose A.

ST7 Points chosen: F, B, D, E, A.

ST8 Since goal only concerns angles, this step does not apply.



Since AB is not parallel to CD, CDA' is a well formed triangle.

By RR1, AE = A'E' = EC.

By RR2, AE || A'E' and A'E' || EC.

So ECA'E' is a parallelogram.

Therefore CA' || EE'.

By RR1, AB = A'B' = CD.

So triangle CDA' is isosceles and angle DCA = angle DA'C.

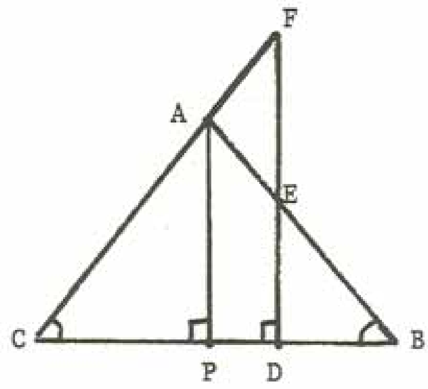
Now since CA' || EE', the angle CD makes with EF is the same as the angle A'B' makes with EF.

By RR2, AB || A'B'.

So the angle AB makes with EF is the same as the angle A'B' makes with EF.

Then by transitivity, the goal is proved.

EXAMPLE 5



GIVEN: AC = AB. D is any point of BC, DEF is perpendicular to BC, AP is perpendicular to BC

PROVE: 2(AP) = (DE + DF)

Let us try reflection.

ST1 P is the midpoint of BC. Choose P, C, B.

ST2 AP intersects P. Choose A.

ST3 DNA

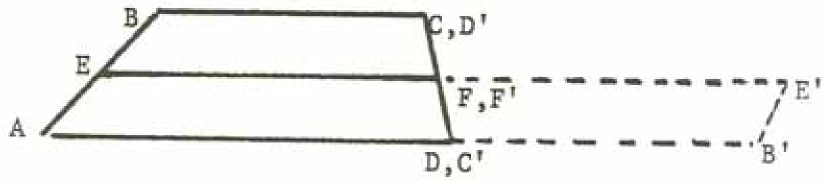
ST4 DNA

ST5 AP is perpendicular to BC so we reject A.

ST6 We have no more points to choose in ST2. Also there are no more midpoints to reflect around. So we will give up trying reflection on this problem.

The next examples are additional problems to which the Midpoint Reflection Construction can be applied.

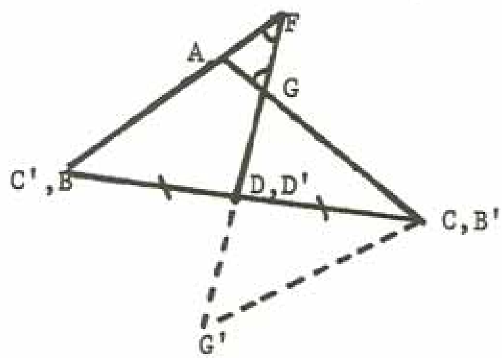
EXAMPLE 6



GIVEN: trapezoid ABCD, AD is parallel to BC, AE = EB, CF = FD

PROVE: EF || AD || BD, EF = $\frac{1}{2}$(AD + BC)

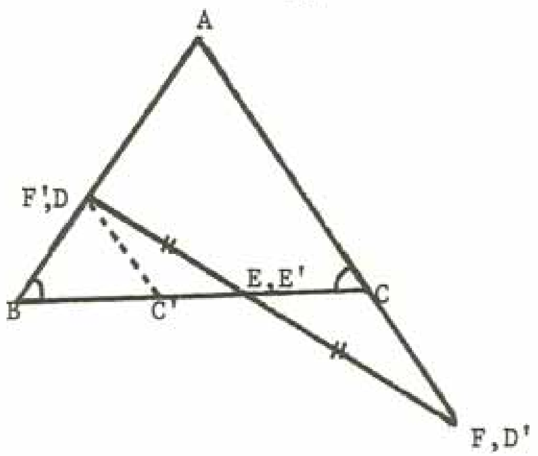
EXAMPLE 7



GIVEN: AC > AB, BD = DC, angle AFG = angle AGF

PROVE: AF = $\frac{1}{2}$(AC - AB)

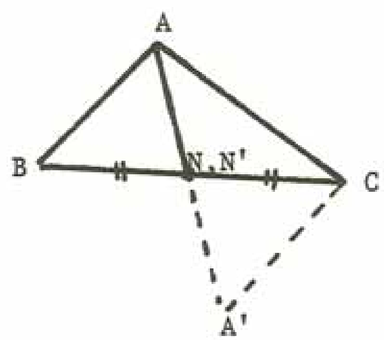
EXAMPLE 8



GIVEN: AC = AB, D is any point on AB, DE = EF

PROVE: CF = BD

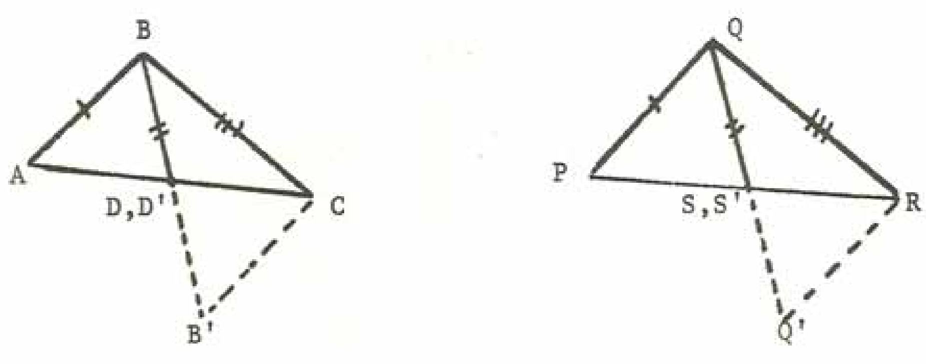
EXAMPLE 9



GIVEN: angle BAM = angle MAC, BM = MC

PROVE: AB = AC

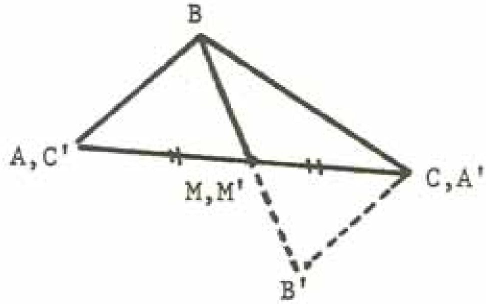
EXAMPLE 10



GIVEN: AB = PQ, BD = QS, BC = QR

PROVE: triangle ABC ≅ triangle PQR

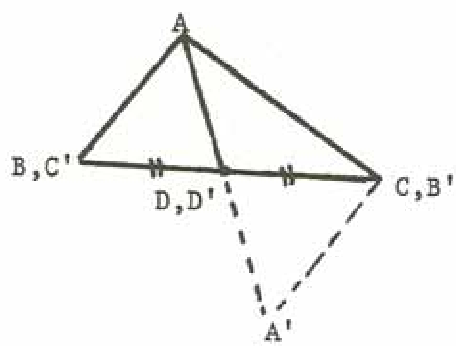
EXAMPLE 11



GIVEN: AM = MC

PROVE: BM < $\frac{1}{2}$(AB + BC)

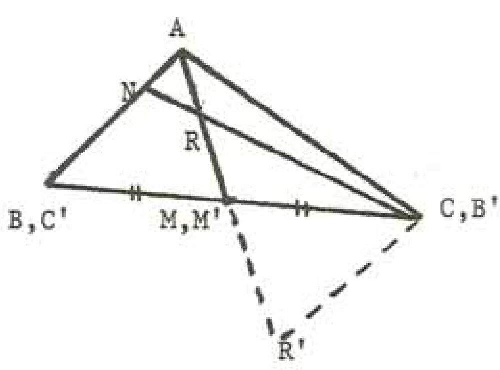
EXAMPLE 12



GIVEN: angle BAC is acute, BD = DC

PROVE: 6(AD) > (AB + AC + BC)

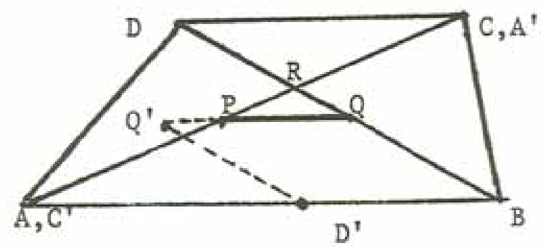
EXAMPLE 13



GIVEN: BM = MC, AN = $\frac{1}{3}$ AB

PROVE: AR = RM, MR = $\frac{1}{4}$ MC

EXAMPLE 14



GIVEN: trapezoid ABCD, AB || CD, P is the midpoint of AC, Q is the midpoint of BD

PROVE: PQ || DC || AB

2.2 Situational Construction Heuristics

This section contains a description of the Situational Construction Heuristics (SCH's). As the name suggests, these heuristics generate constructions for certain local situations in geometry. Altogether, six construction situations will be given in the description of the SCH's.

The format of the description consists of five main parts. The first two parts are a description of the construction situation. The situation is described in terms of the constraints on the figure and the problem goal. The third part specifies the construction that should be applied. The fourth part contains the motivation for drawing the specified construction. The fifth part contains example problems in which the SCH can be used.

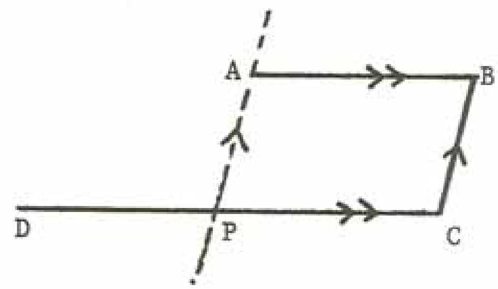
SCH1 Situation: AB $||$ CD, AB $\neq$ CD

Goal: to prove an equality of the form

$$AB + X_1Y_1 + \ldots + X_NY_N = CD + R_1S_1 + \ldots + R_MS_M ,$$

$$N \geq 1,\ M \geq 1,\ X_1Y_1 \neq 0$$

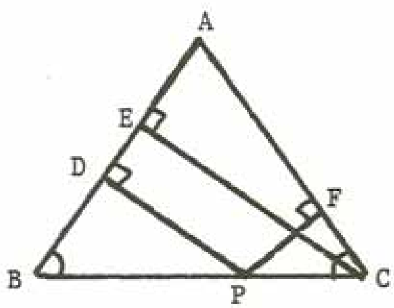
Construction: From A draw a line parallel to BC intersecting CD at P.



Motivation: In general there are 2 goals which a construction should achieve. One is that it should somehow bring us closer to a solution. The other is that it should bring the solution closer through the utilization of the constraints on the figure. This is a vital goal since a complete proof will utilize all the constraints on the figure (unless the problem is overspecified). So a construction should allow us to make use of a constraint in proving the goal. SCH1 creates a parallelogram which proves AB equal to part of CD. Since this equality is part of the problem goal, we are closer to the solution. Also in our proof of the equality, we utilized the constraint that AB $||$ CD. Therefore both goals of the construction have been achieved. Another reason for making this construction is that the usual methods of geometry (such as congruency) are only able to prove equality between 2 pairs of line segments. The methods are not helpful in proving equality about a sum of line segments. So this construction helps to reduce the form of the goal in in that it eliminates a sum. Hopefully, this construction will also be able to reduce the goal to proving a single pair of segments equal.

EXAMPLES OF SCH1

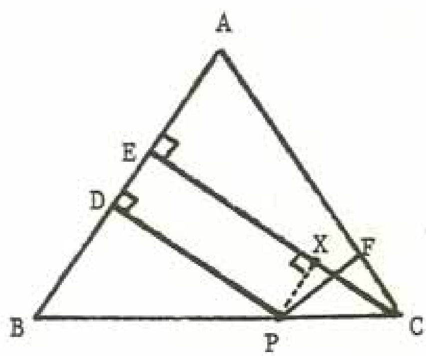
EXAMPLE 15



GIVEN: PD ⊥ AB, EC ⊥ AB, PF ⊥ AC, AB = AC

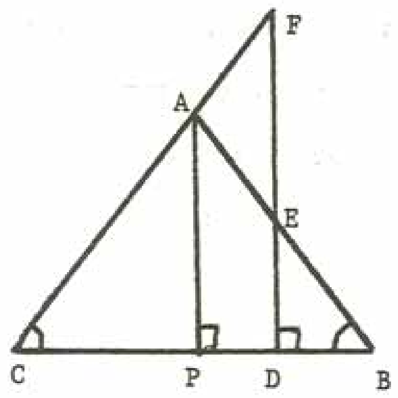
PROVE: EC = (DP + PF)

DP || EC and DP ≠ EC, the goal is the correct form. Using SCH1 draw AX || PD.



The problem can now be solved.

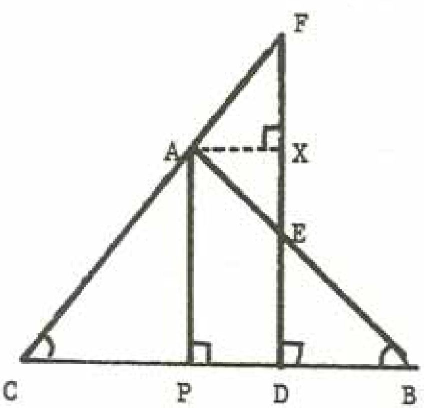
EXAMPLE 16



GIVEN: AC = AB, AP ⊥ BC, FD ⊥ BC

PROVE: 2(AP) = (DE + DF)

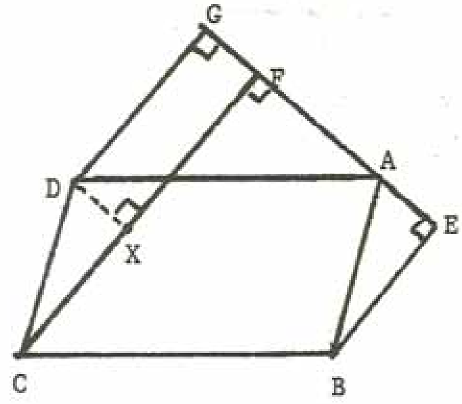
AP || DF, AP ≠ DF, the goal is of the correct form. Using SCH1 draw AX || PD.



The problem can now be solved.

The next 5 examples are additional problems to which SCH1 can be applied.

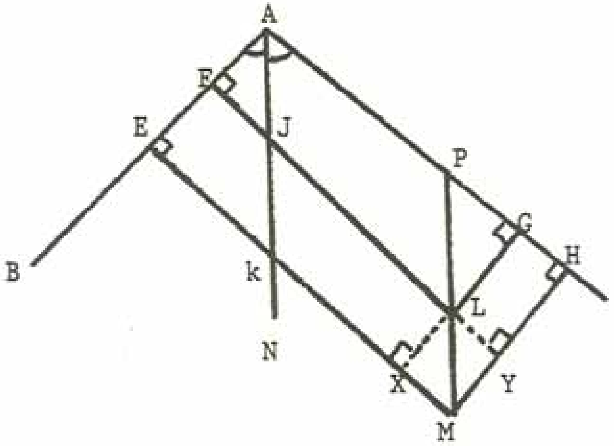
EXAMPLE 17



GIVEN: parallelogram ABCD, CF ⊥ EG, DG ⊥ EG, BE ⊥ EG

PROVE: CF = (BE + DG)

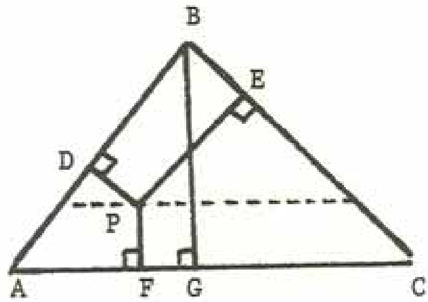
EXAMPLE 18



GIVEN: angle BAN = angle NAC, LF ⊥ AB, ME ⊥ AB, LG ⊥ AC, MH ⊥ AC, MD || AN

PROVE: (FL - GL) = (EM - MH) or (FL + MH) = (EM + GL)

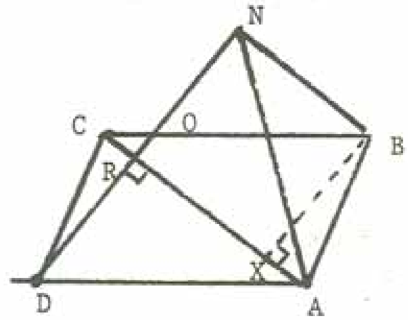
EXAMPLE 19



GIVEN: equilaterial triangle ABC with P any point within ABC. DP, EP, and FP are perpendicular to AB, BC, AC; BG ⊥ AC

PROVE: BG = (PD + PE + PF)

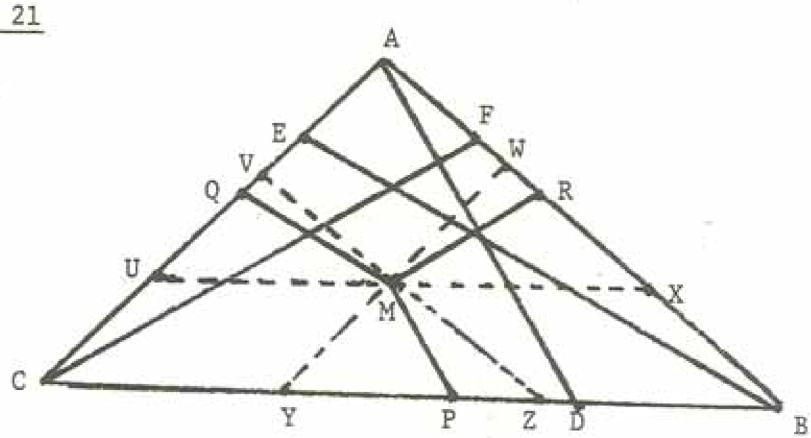
EXAMPLE 20



GIVEN: parallelogram ABCD, DN $\perp$ AC, BN $||$ AC

PROVE: AR = CR + BN

EXAMPLE 21



GIVEN: AD = BE = CF, MR $||$ CF, MQ $||$ BE, MP $||$ AD

PROVE: AD = BE = CF = (MP + MQ + MR)

Before we state the next SCH, it is necessary to make a definition.

DEFINITION: P is a ratio point if P lies on a line XY, P between X and Y, and if the ratio between XP and PY is an important ratio in the problem. (XP|PY) is an important ratio if it is part of a constraint on the problem or if the ratio is part of the problem goal.

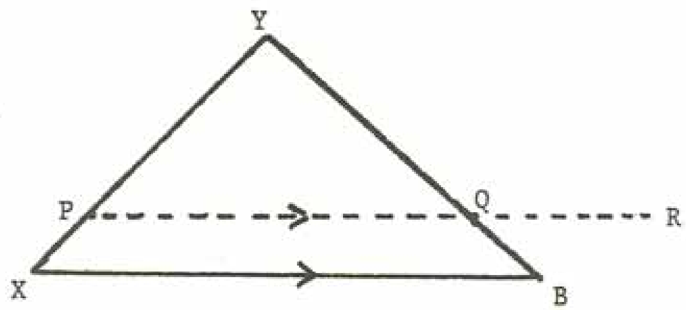
Examples of ratio points are: midpoints (the ratio $\frac{XP}{PY}$ is constrained to be 1), points P on a line XY when the goal is of the form $\frac{XP}{PY} = \frac{AB}{CD}$, and points P on a line XY when the ratio $\frac{XP}{PY}$ is determined by the ratio of 2 other segments in the problem ($\frac{XP}{PY} = \frac{AB}{CD}$).

Now, making use of the above definition, we can define the next SCH.

SCH2 Situation: there is a ratio point P on a line XY in the figure

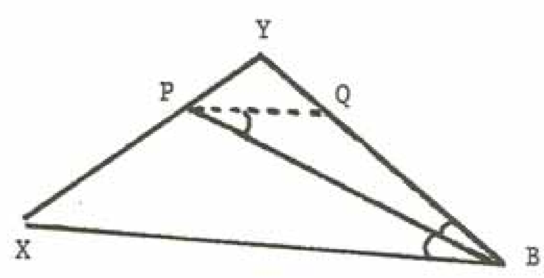
Goal: the goal may be anything

Construction: Choose a line XB which is not colinear with XY. From P draw a line parallel to XB that intersects YB at Q. If the goal of the problem is to prove a relationship between various segments of the figure, try to make XB one of the segments in the goal.



Motivation: The general motivation for this construction is that after it has been performed, the theorem "a line parallel to one side of a triangle and intersecting the other two sides divides these sides into propertial segments" can be applied (i.e., since PQ || XB, $\frac{YP}{PX} = \frac{YQ}{QB}$).

More specifically, there are 2 different situations which contain ratio points in which this construction is especially useful. The first situation is when XB is one of the segments of the goal or when the ratio $\frac{XP}{PY}$ is part of the goal. Then the construction will create a new representation of the goal. PQ is some fraction of XB and the ratio $\frac{YQ}{QB}$ is equal to $\frac{XP}{PY}$. This new representation of the goal can make the problem solution much easier. For example, if the problem goal is to prove $\frac{1}{2}$(XB) equal to some other segment and if there is no segment of length $\frac{1}{2}$(XB) in the figure, then to deduce the goal using the normal congruency methods of geometry will usually be difficult. If P is the midpoint of XY, then this construction will create a segment of length $\frac{1}{2}$(XB). This will allow us to use the normal congruency methods of geometry to prove PQ = $\frac{1}{2}$(XB) = GH. The deduction of PQ = GH should be much easier than the goal $\frac{1}{2}$(XB) = GH. The second specific situation is when PB is the angle bisector of angle YBX.

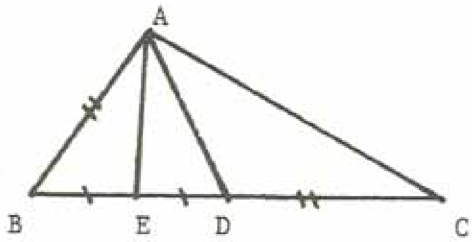


The construction allows us to transform the constraint of angle YBP = angle PBY into one of 2 equal segments, PQ and

QB. The constraint that a line is an angle bisector is sometimes difficult to utilize in a proof. By transforming the constraint it may be easier to utilize in a proof.

EXAMPLES OF SCH2

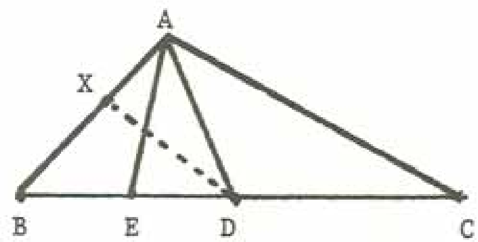
EXAMPLE 22



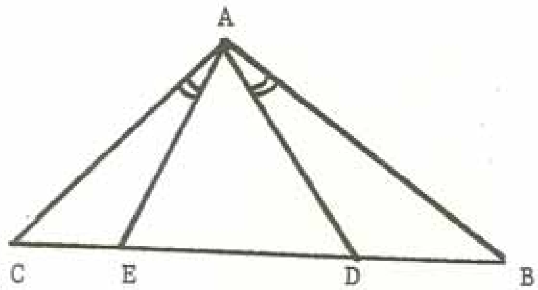
GIVEN: BC = 2AB, BE = ED, BD = DC

PROVE: angle EAD = angle DAC

E and D are both ratio points (they are both midpoints). Using SCH2 draw DX || AC. The problem can now be solved.



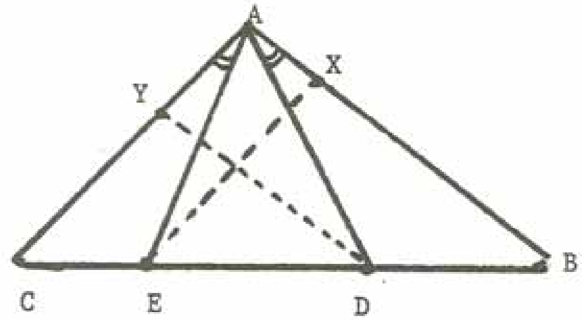
EXAMPLE 23



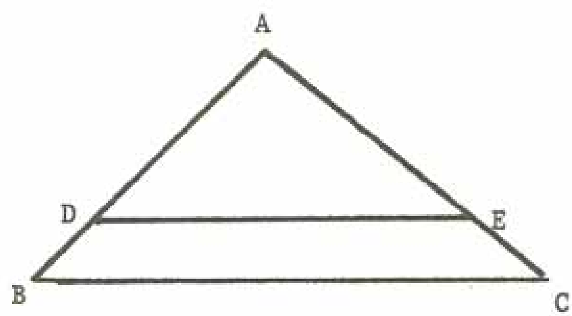
GIVEN: angle CAE = angle DAB

PROVE: $\frac{(AB)^2}{(AC)^2} = \frac{(BD)(BE)}{(CE)(CD)}$

E and D are both ratio points since the ratios $\frac{BD}{DC}$ and $\frac{BE}{EC}$ are parts of the goal. Using SCH2 draw EX and DY parallel to AC and AB. Note that AB and AC are segments of the goal.



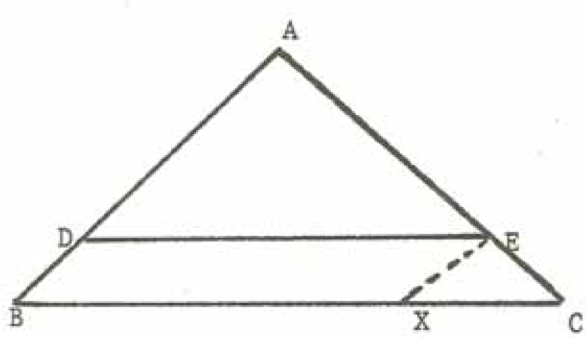
EXAMPLE 24



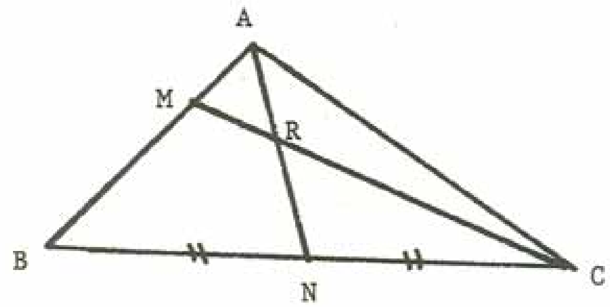
GIVEN: $\frac{AD}{BD} = \frac{AE}{EC}$

PROVE: DE || BC

D and E are both ratio points since the ratios $\frac{AD}{BD}$ and $\frac{AE}{EC}$ are part of constraints of the problem. Using SCH2 draw EX || AB.



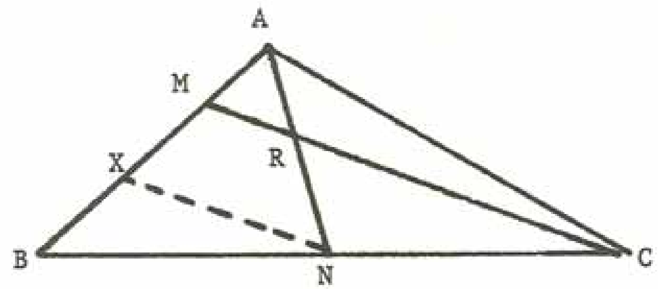
EXAMPLE 25



GIVEN: BN = NC, AM = $\frac{1}{3}$ (AB)

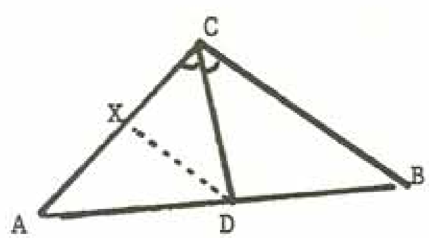
PROVE: AR = RN, MR = $\frac{1}{4}$(MC)

N is a ratio point since it is a midpoint. MR is a segment of the goal. Using SCH2 draw NX || MR.



The next 3 examples are additional problems to which SCH2 can be applied.

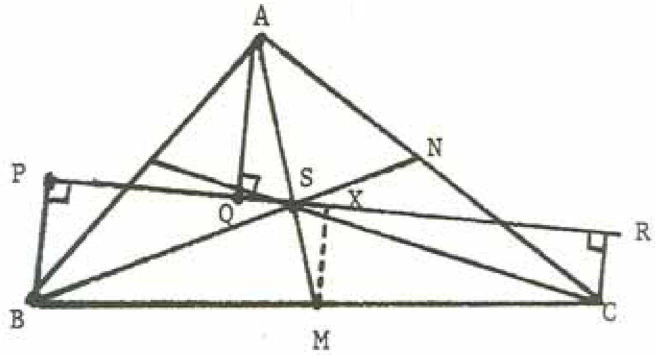
EXAMPLE 26



GIVEN: angle ACD = angle DCB

PROVE: AD/DB = AC/CB

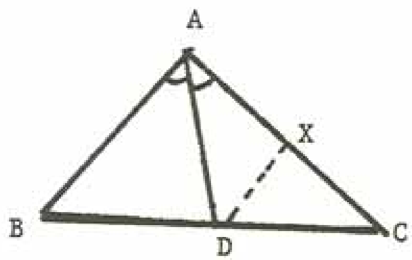
EXAMPLE 27



GIVEN: AN = NC, BM = MC, BP ⊥ PR, CR ⊥ PR, AQ ⊥ PR

PROVE: AQ = (BP + CR)

EXAMPLE 28



GIVEN: AB > AC, angle BAD = angle DAC

PROVE: AD < $\frac{1}{2}$(AB + AC)

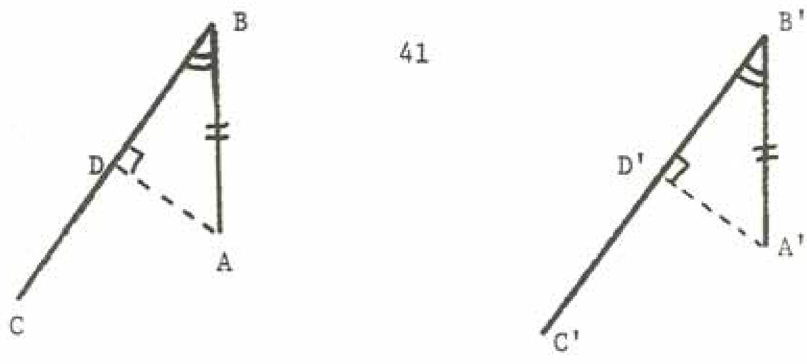
For additional problems to which SCH2 can be applied, see the problems in examples 1, 3, and 7-12.

SCH3 Situation: AB = A'B', angle ABC = angle A'B'C', and these 2 constraints are not corresponding parts of some pair of congruent triangles.

Goal: to prove some kind of segment or angle equality.

Construction: 1) draw AD ⊥ BC

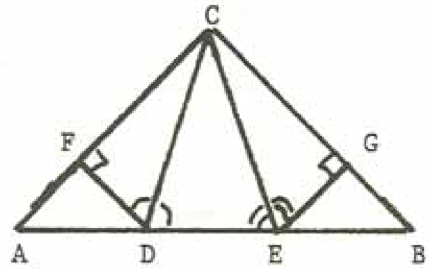
2) draw A'D' ⊥ B'C'



Motivation: The basic motivation of this heuristic is to creat a congruency in the figure. If this congruency solves part or all of the problem, then the construction has achieved its purpose. If this construction creates new triangles that contain elements of the goal, it is intended that the congruency created by the construction will enable us to prove another congruency which will involve elements of the goal.

EXAMPLES OF SCH3

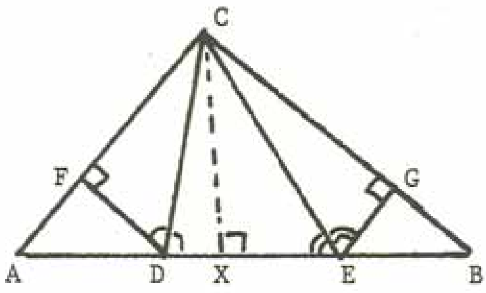
EXAMPLE 29



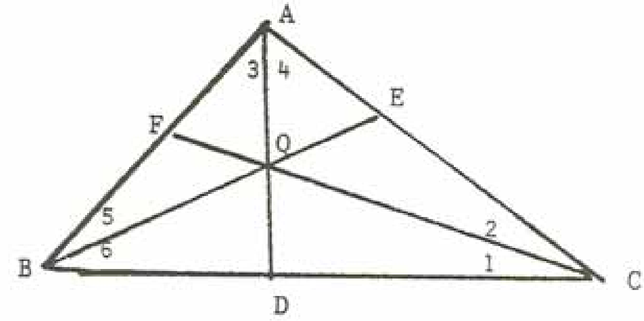
GIVEN: angle FDC = angle EDC, angle DEC = angle CEG, DF ⊥ AC, EG ⊥ BC

PROVE: DE = (FD + GE)

Since angle FDC = angle EDC, CD = CD, and the goal involves segment equality, we can apply SCH3 and draw CX perpendicular to AB.



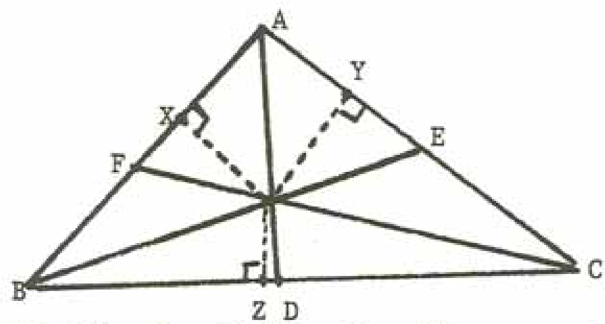
EXAMPLE 30



GIVEN: angle 5 = angle 6, angle 1 = angle 2

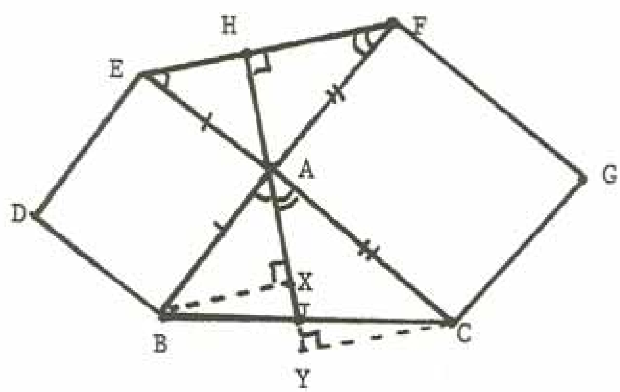
PROVE: angle 3 = angle 4

The goal involves an angle equality so we can use SCH3. Since angle 5 = angle 6 and QB = QB, draw QX ⊥ AB and QZ ⊥ BC using SCH3. Again using SCH3, since angle 1 = angle 2 and OC = OC draw QY ⊥ AC.



The next 6 examples will be additional problems to which SCH3 can be applied.

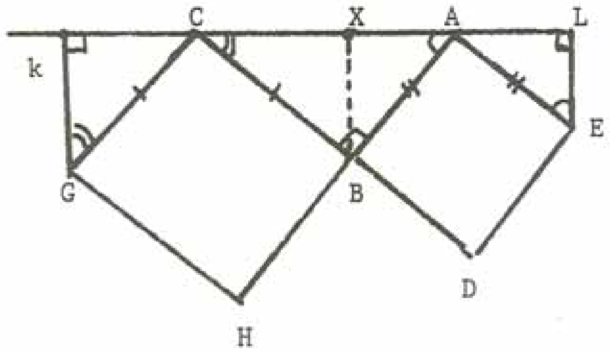
EXAMPLE 31



GIVEN: ABDE and ACGF are squares, IH ⊥ EF

PROVE: BI = IC

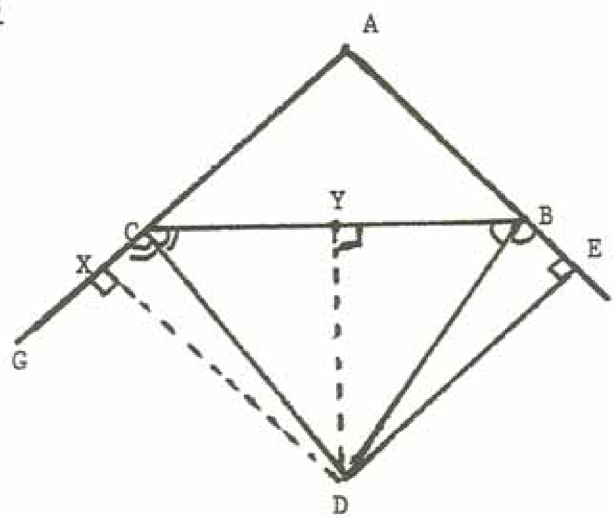
EXAMPLE 32



GIVEN: AB ⊥ BC, BHGC and ABDE are squares, LE ⊥ AL, GK ⊥ KC

PROVE: AC = (EL + GK)

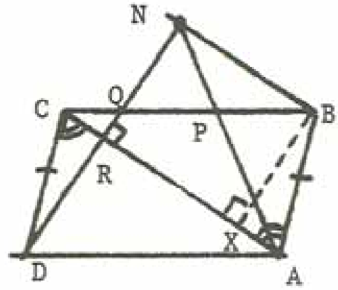
EXAMPLE 33



GIVEN: angle DCG = angle DCB, angle DBC = angle DBE, DE ⊥ AE

PROVE: AE = $\frac{1}{2}$(AC + BC + AB)

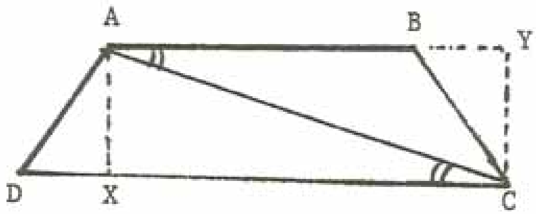
EXAMPLE 34



GIVEN: parallelogram ABCD, DN ⊥ AC, BN || AC

PROVE: PQ = PB

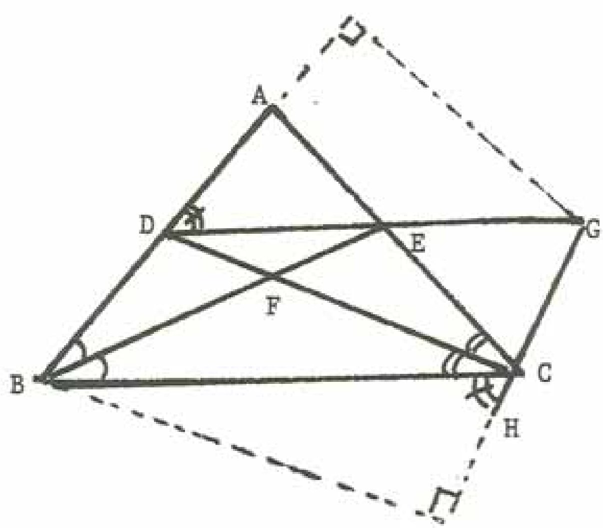
EXAMPLE 35



GIVEN: trapezoid ABCD, AB || CD, angle ADC = angle BCD

PROVE: AD = BC

EXAMPLE 36



GIVEN: BE = CD, BE bisects angle ABC, CD bisects angle ACB, angle ABE = angle GDC, angle ADG = angle BCH, BC = DG

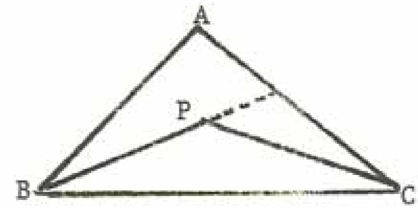
PROVE: AB = AC

SCH4 Situation: an intersection point of 2 or more line segments is constrained to lie within a triangle

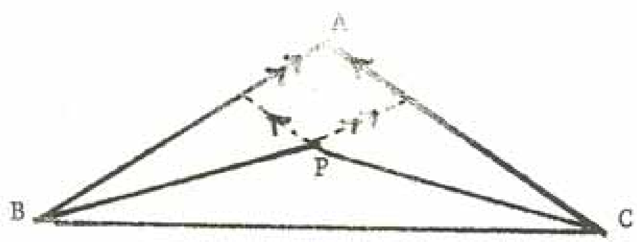
Goal: to prove either an angle inequality or a segment inequality

Construction: The principle objective is to draw a line from the intersection point P to a side of the triangle. This can be done in either of 2 ways:

1) extend one of the line segments which help form P so that the segment will meet one of the sides of the triangle.



2) from the intersection point P draw lines parallel to the sides of the polygon.



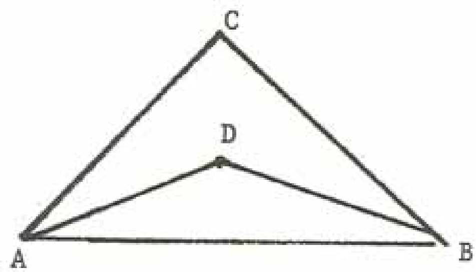
Motivation: Since the goal involves an inequality one method of solving the problem would be to construct some new triangles so that the triangle inequality theorem or the exterior angle theorem could be applied to solve the goal.

The construction described above creates new triangles so that the inequality theorems may be applied. Also by drawing lines through P, a way of utilizing the constraint that

P is inside the triangle is provided. The triangles created by the construction would have different relationships with the other parts of the figure if P was outside the triangle.

EXAMPLES OF SCH4

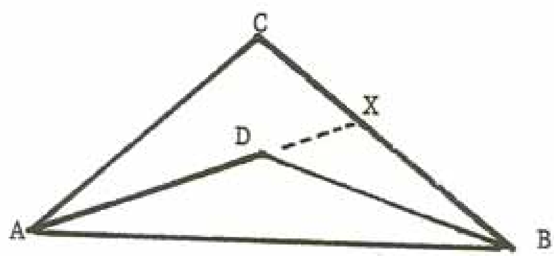
EXAMPLE 37



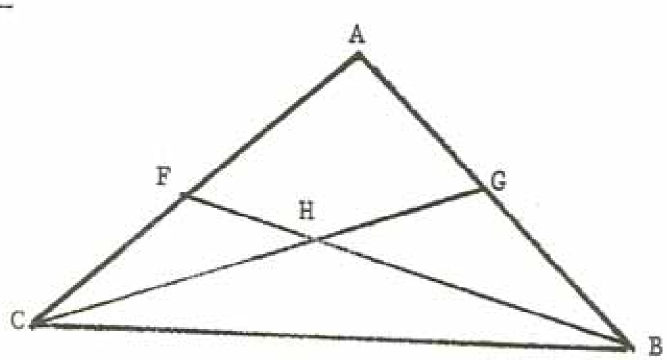
GIVEN: triangle ABC with the point D within ABC

PROVE: angle ADB > angle ACB

D is constrained to be within ABC. The goal is an angle inequality. Because of these conditions we can apply SCH4 and extend D so that it intersects BC at X.



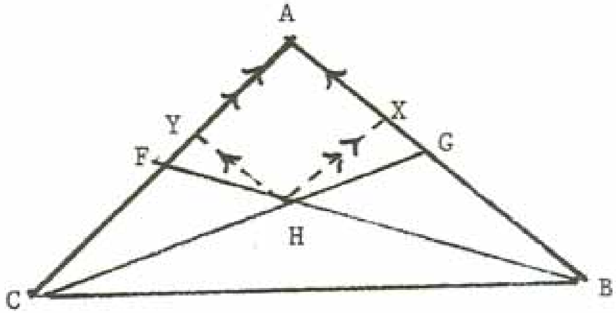
EXAMPLE 38



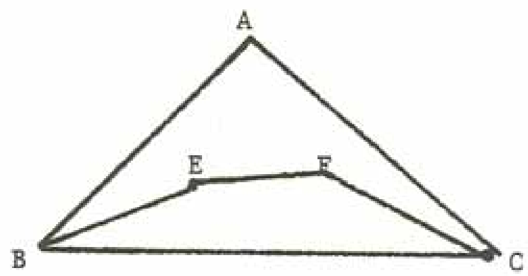
GIVEN: FB and CG are any two lines drawn from C and B that intersect inside ABC

PROVE: (AF + AG) > (HF + HG)

H is constrained to be within ABC. The goal is a segment inequality. Because of these conditions we can apply SCH4 and draw HX || AF and HY || AG.



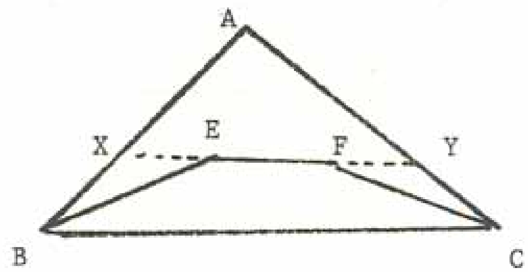
EXAMPLE 39



GIVEN: points E and F are within triangle ABC

PROVE: (AB + AC) > (BE + EF + FC)

E and F are constrained to be within ABC. The goal is a segment inequality. Because of these conditions we can apply SCH4 and extend FE so that it intersects AB at X and extend EF so that it intersects AC at Y.



SCH5 Situation: AB = 2CD and the midpoint of AB is not the intersection of 2 or more lines

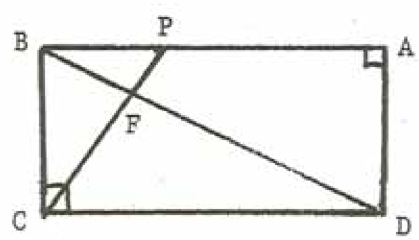
Goal: to prove anything

Construction: Consider the midpoint of AB to be the intersection of 2 or more lines. Then try to apply the normal construction heuristics of geometry. Note that the trivial construction of connecting 2 points can be applied to the midpoint of AB.

Motivation: The normal proof methods of geometry (such as congruent triangles) deal only with equal segments and angles. The constraint AB equals 2CD is not in this form so it may be difficult to utilize this constraint in a proof. With the above construction we have converted the constraint into one concerning equal segments. This should make the formulation of a proof easier.

EXAMPLES OF SCH5

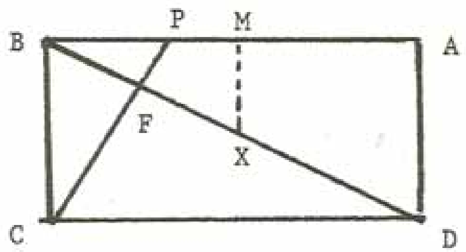
EXAMPLE 40



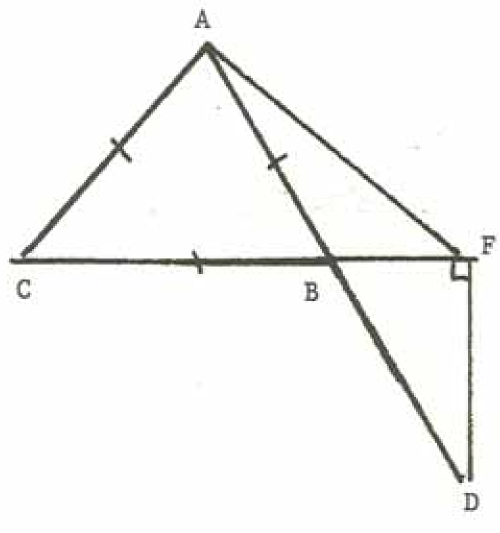
GIVEN: rectangle ABCD, AB = 2BC, BP = $\frac{1}{4}$(AB)

PROVE: BD $\perp$ CP

We are given that AB = 2BC. So using SCH5 we consider M, the midpoint of AB, to be the intersection of 2 or more lines. Then using SCH2 we can draw MX || BC.



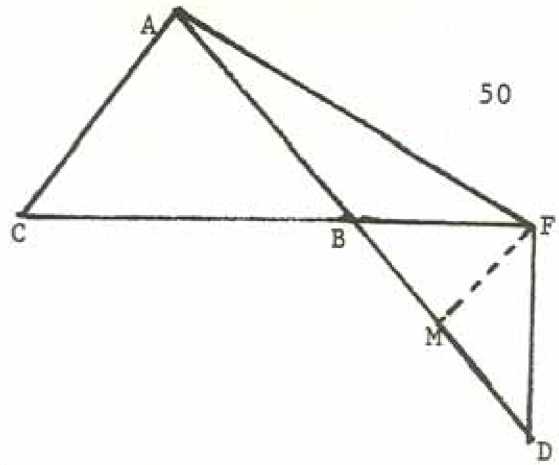
EXAMPLE 41



GIVEN: AC = AB = BC, BD = 2AB, FD $\perp$ FC

PROVE: angle FAC is a right angle

We are given that BD = 2AB. So using SCH5 we consider M, the midpoint of BD, to be the intersection of 2 or more lines. Then the problem can be solved by drawing the segment MF.

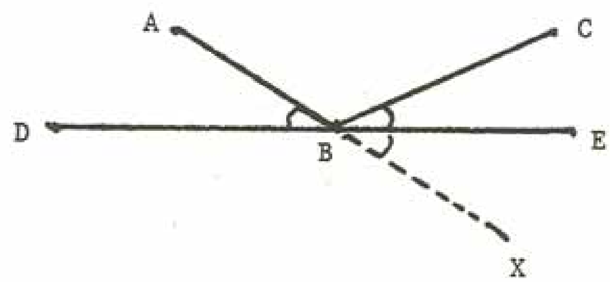


The problem contained in example 25 can also be solved by using SCH5.

SCH6 <u>Situation</u>: angle ABD = angle CBE, DBE is a line, A and C are on the same side of DBE

<u>Goal</u>: to prove an equality or inequality with one of the terms being the sum (AB + BC), e.g., (AB + AC) = PQ

<u>Construction</u>: Extend AB through B a distance equal to BC.

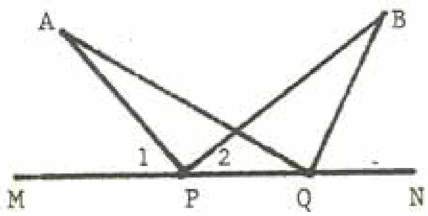


<u>Motivation</u>: Through the construction we have created a single segment AX equal to the sum of (AB + BC). A proof using the regular triangle methods of geometry is much easier to create if the elements of the goal are single segments like AX instead of sums of segments like (AB + BC). Also the constraing of angle ABD = angle CBE can more easily be utilized in proving a pair of congruent triangles. Since angle XBE =

angle ABD = angle CBE, triangle CBE is congruent to triangle BXE. This congruency can be very useful in proofs. See the examples which follow.

EXAMPLES OF SCH6

<u>EXAMPLE 42</u>

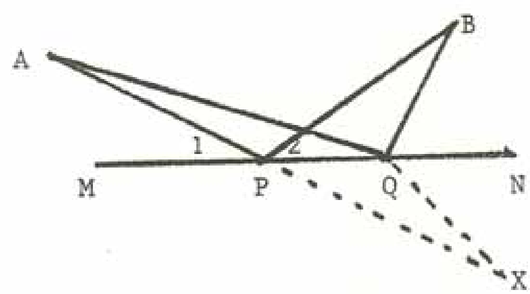


GIVEN: A and B are on the same side of straight line MN, angle 1 = angle 2,

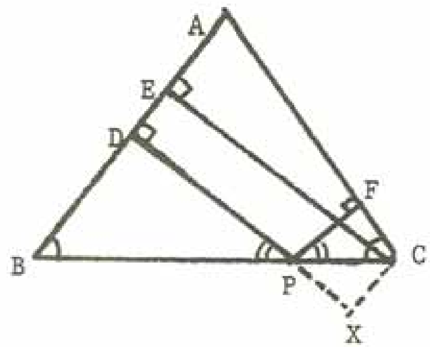
Q is any other point on MN

PROVE: (AP + BP) < AQ + BQ

The goal involves an equality with the term (AP + BP). Angle 1 = angle 2. Because of these conditions we can apply SCH6 and extend AP through P for a distance equal to BP.



EXAMPLE 43



GIVEN: PD ⊥ AB, EC ⊥ AB, PF ⊥ AC, AB = AC

PROVE: EC = (DP + PF)

The goal involves an equality with the term (DP + PF). It can be deduced that angle DPB = angle FPC. Because of these conditions we can apply SCH6 and extend DP through P for a distance equal to FP.

2.3 Constructions to Apply to a Previously Proven Theorem

This section will discuss heuristics to create constructions that enable the application of a previously proven theorem. This type of construction is really very similar to the constructions described in the SCH section. Many of the SCH constructions can be considered to be constructions that allow a previously proven theorem to be applied. The only difference is the type of theorem that the SCH's utilize. The SCH's apply very simple basic theorems such as the side angle side congruency theorem, or the theorem that the opposite sides of a parallelogram are parallel. Also, the opportunity to apply these simple theorems occurs quite frequently in geometry.

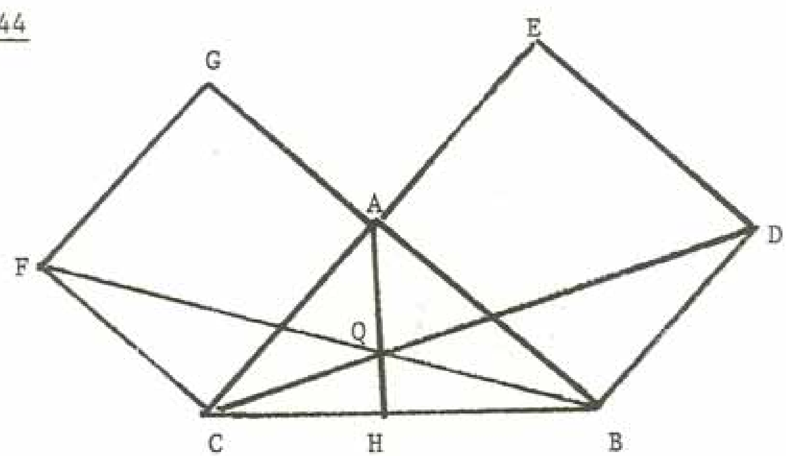
In contrast, the theorems which will be discussed in this section are more complicated. They are not the basic theorems of geometry. Because of their complexity, situations for the useful application of these theorems does not occur as frequently as for the basic theorems of the SCH's.

The first step in drawing this type of construction is to choose an appropriate theorem to apply. This decision is usually based on whether the results of the theorem are useful for the solution of the problem. For example, the theorem that the altitudes of a triangle are concurrent would be considered useful if the goal of the problem was to prove that two lines were perpendicular to each other. Another consideration would be whether any of the constraints of the theorem were present in the problem. For example, we would check for perpendiculars in the figure if we wanted to apply the theorem that the altitudes of a triangle are concurrent.

Once a theorem has been chosen, the next step is to draw the needed constructions. The constructions can be drawn with this one simple heuristic: establish a correspondence between the points of the theorem figure and the points of the problem. The parts of the theorem figure that have no counterparts in the problem figure are the parts which must be inserted into the problem figure by construction. This correspondence of the figures can usually be established by pairing the elements of the theorem goal with the elements of the problem goal. Another method of establishing this correspondence is to pair the constraints of the problem with the constraints of the theorem.

Some examples of the use of this heuristic will now be given.

EXAMPLE 44

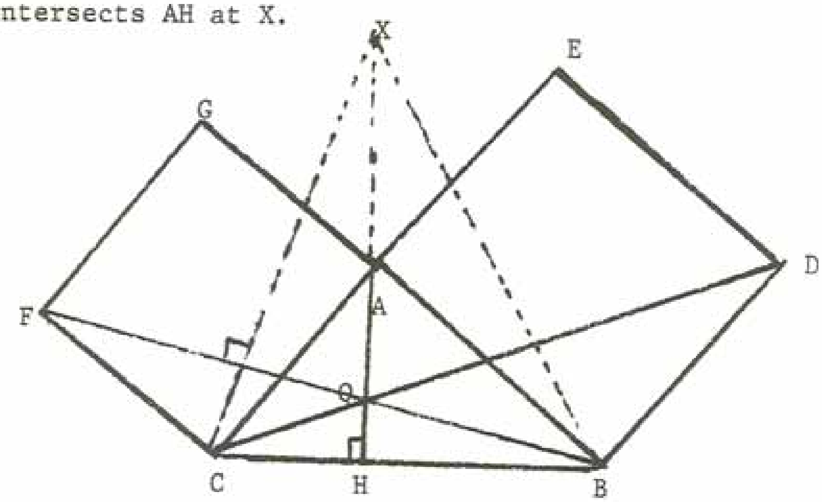


GIVEN: ABDE and ACFG are squares, AH ⊥ BC

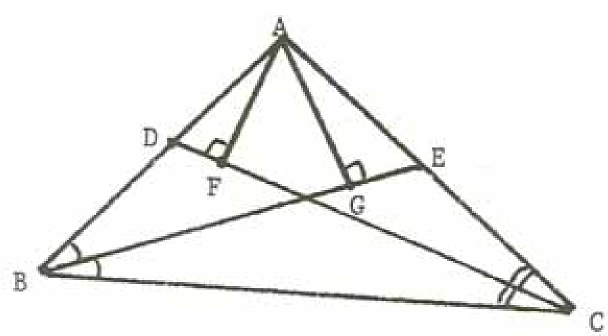
PROVE: FB, DC, and AH are all concurrent at a point Q

The theorem to be applied is the theorem that the altitudes of a triangle are concurrent. By mathcing the elements of the problem goal and the theorem goal, it can be seen that we wish to create a triangle which has parts of the lines FB, CD, and AH as its altitudes. It is not possible to construct such a triangle immediately. However, we can achieve most of the matchings by drawing a triangle which has parts of the lines FB and AH as its altitudes. Also, part of CD is a line in the interior of the constructed triangle.

Construction: AH is extended through A. CX is perpendicular to FB and intersects AH at X.



EXAMPLE 45

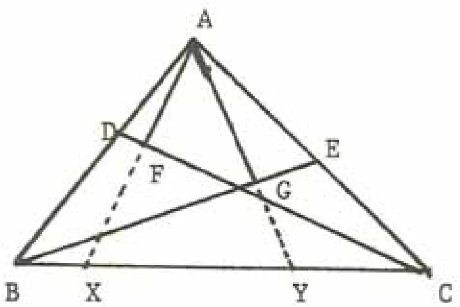


GIVEN: BE bisects angle ABC, DC bisects angle ACB, AF ⊥ DC, AG ⊥ BE

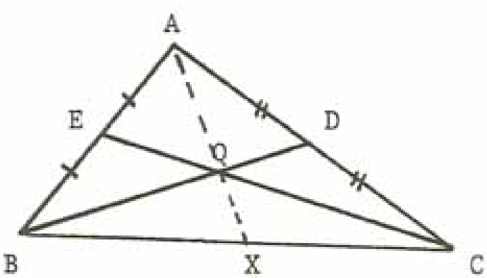
PROVE: FG || BC

The theorem that the line connecting the midpoints of 2 sides of a triangle is parallel to the third side is the theorem to be applied. By matching the goal elements we can see that we need a triangle that has part of BC as its base and part of FG as a line connecting the midpoints of the 2 other sides.

<u>Construction</u>: draw FX ⊥ FC and GY ⊥ BG



<u>EXAMPLE 46</u>



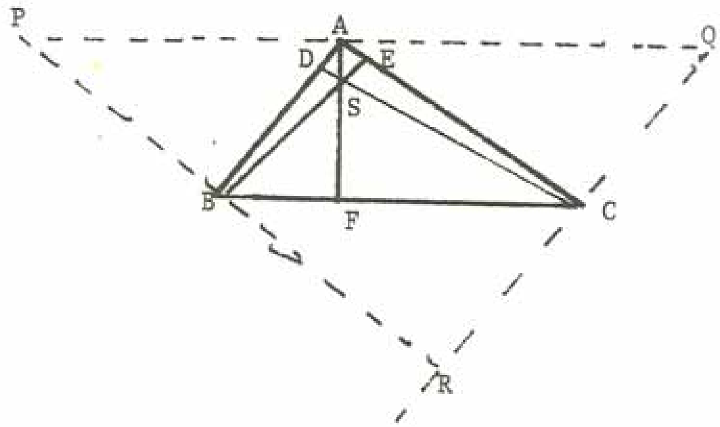
GIVEN: BE = EA, AD = DC, AC > AB

PROVE: EC > BD

Theorems to be applied: 1) the medians of a triangle are concurrent, 2) if two triangles have two sides respecitvely equal to two sides of the other, and the third sides unequal, then the angle contained by the sides of that with the greater base is greater than the corresponding angle of the other.

<u>Construction</u>: draw AX the median of BC

EXAMPLE 47



GIVEN: triangle ABC with altitudes AF, BE, and CD

PROVE: AF, BE, and CD meet at a point

Theorem to be applied: The perpendicular bisectors of the sides of a triangle are concurrent.

Constructions: draw PQ so that PA = AQ = BC and PQ $\perp$ AF, draw PR so that PB = BR = AC and PR $\perp$ BE, draw QR so that AB = RC = CQ and CD $\perp$ RQ.

2.4 Evaluation of the Construction Heuristics

The 3 types of construction heuristics described are very effective for most of the simple problems in geometry which require constructions. By simple problems it is meant problems of elementary and intermediate difficulty for a student who has a good knowledge of high school geometry.

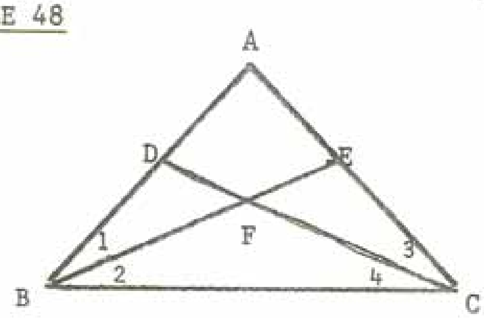
The midpoint reflection heuristic, SCH1, SCH2, and SCH3 have a wide range of applicability. Each heuristic can be successfully applied to a number of problems. SCH4, SCH5, and SCH6 are more limited in their applicability. They are designed to handle special situations in which the more general heuristics are not applicable. In applying all these heuristics to a problem, this group of heuristics should be applied only after the more general heuristics have been applied.

The constructions to apply a previously proven theorem are usually easy to generate. The procedure of identifying points and lines of the problem figure with the theorem figure is very simple and effective. Although this type of construction is quite simple from the point of view of actually generating the points and lines, this problem solving technique does have one major difficulty. The problem of deciding which theorem to apply can be very difficult. In a complex problem figure, a large number of theorems may be applicable.

The above three types of heuristics are all similar in one respect. They all essentially identify some local situation in geometry and through this identification decide which construction should be drawn. This type of heuristic is very effective for the simple problems in geometry. For the difficult geometry problems (such as the internal angle bisector problem), however, this type of heuristic is not effective. Although the local heuristics can be applied to these hard problems and constructions generated, the problems cannot be solved with these constructions. This failure is due to global conditions in the problem which the construction heuristics are unable to deal with.

Some examples of these difficult problems will be given. Because of the scarcity of this type of problem, there will be no attempt to give heuristics for the solution of these difficult problems. With such a small sample of problems it is very difficult to make any deductions about the general properties of these problems.

EXAMPLE 48



(THE INTERNAL ANGLE BISECTOR PROBLEM)

GIVEN: angle 1 = angle 2, angle 3 = angle 4, DC = BE

PROVE: AB = AC

First, we can apply SCH3 to the problem. Since angle 1 = angle 2, BE = BE, and BF = BF we can draw perpendiculars from D, F, and E.

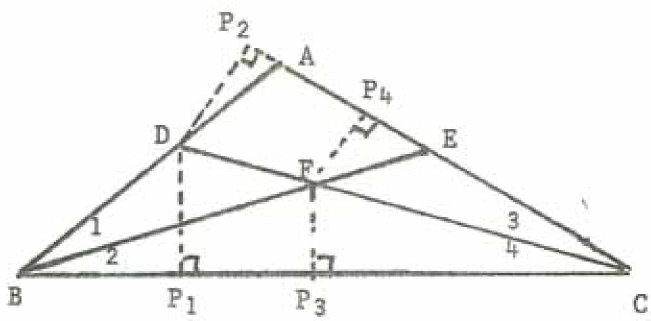


Fig. 7

In Fig. 7 , using SCH3, we have drawn DP_2 and FP_4 perpendicular to AC and DP_1 and FP_3 perpendicular to BC. Also, we could have drawn the corresponding set of perpendiculars from F and E.

None of the constructions generated through SCH3 is effective in solving the problem. Every construction fails to provide a way to utilize the constraint that DC = BE. The constructions all concentrate on the angle constraints while neglecting the segment constraint.

We can also apply SCH2 to the problem since the angle bisectors in the problem create ratio points.

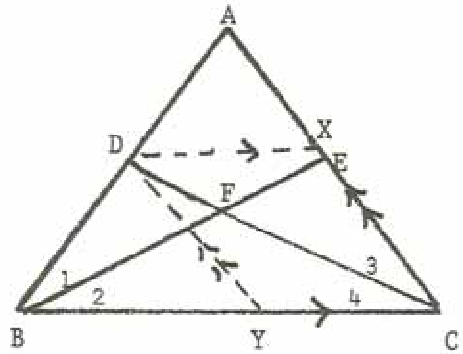
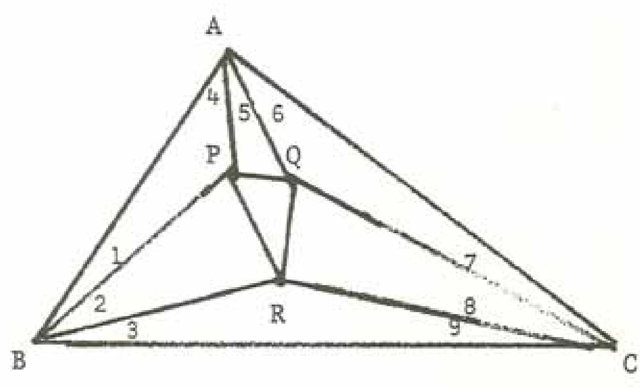


Fig. 8

In Fig. 8, using SCH2, we have drawn DX || BC and DY || AC. Also we could have drawn the corresponding set of parallel from E.
None of the constructions generated through SCH2 is effective in solving the problem. Again, every construction fails to provide a way to utilize the segment constraint of DC = BE. The angle constraints are the only ones utilized by the constructions.

EXAMPLE 49



(THE THEOREM OF MORLEY)

GIVEN: angle 1 = angle 2 = angle 3, angle 4 = angle 5 = angle 6, angle 7 = angle 8 = angle 9

PROVE: PQR is an equilateral triangle

In this problem there are many instances where SCH3 can be applied.

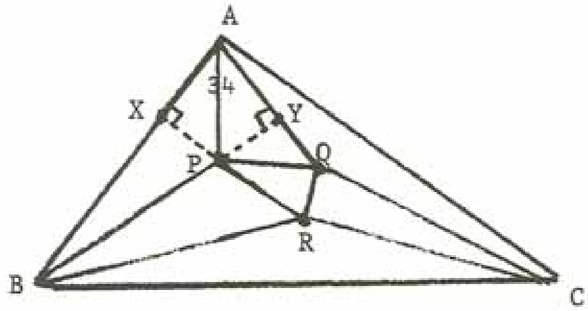


Fig. 9

In Fig. 9, using SCH3, and since angle 3 = angle 4 and AP = AP, we have drawn XP ⊥ AB and PY ⊥ AQ.

We could have also drawn the corresponding set of perpendiculars from AW, CQ, CR, BR and BP.

None of these SCH3 constructions is effective in solving the problem. Each construction considers only a pair of equal angles. The constructions fail to provide a way to utilize in a proof the global constraints of three angles being equal and of all three angle bisectors being present in the triangle.

EXAMPLE 50

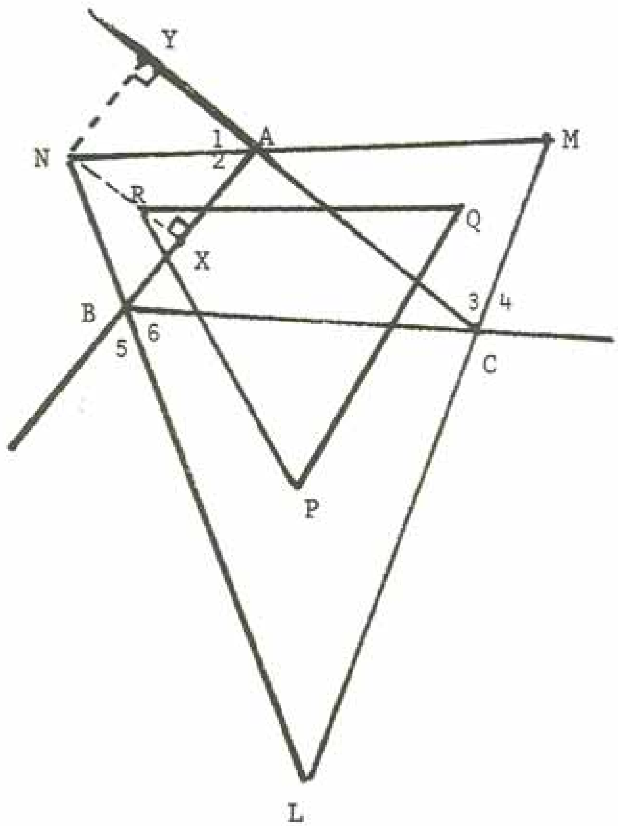


Fig. 10

GIVEN: angle 1 = angle 2, angle 3 = angle 4, angle 5 = angle 6, R, Q, and P are the orthocenters of triangles NAB, MCA, and LBC

PROVE: RQ || BC, RQ = BC

There are three instances in the problem where SCH4 can be applied. In Fig.10, the perpendiculars NX and NY have been added through SCH3. A corresponding set of perpendiculars could also have been added at L and M.

None of these constructions will solve the problem. The constructions are not able to allow us to utilize the constraint that the three angles (angle 1 + angle 2), (angle 3 + angle 4), and (angle 5 + angle 6) are all exterior angles of the same triangle. The constructions only allow us to utilize the constraint of angle equality while they neglect the global constraint of the exterior angles.

3 GEOMETRY PROBLEMS EXPRESSED IN A VECTOR ALGEBRA FORM

3.1 Transformation of Geometry Problems into Vector Algebra Problems

In this half of the thesis we will deal with an algebraic representation of geometry. There will be a discussion of the solutions for geometry problems translated into this algebraic representation. The algebraic form of the geometry construction heuristics described in the first half of this paper will also be discussed.

In order to obtain an algebraic description of geometry problems we will first describe an algebraic representation of geometric figures and relations. There is a simple method for transforming a geometry figure description into a vector algebra description. Consider every point in the geometric figure to be a vector in a two dimensional vector space. For every point in the figure, the corresponding vector can be considered to be a directed line segment from an arbitrary origin to the point. So each point in the geometry figure becomes a variable in the vector algebra description. A line segment is represented as the difference of 2 vectors. These 2 vectors are, of course, the vectors representing the 2 endpoints of the line segment. (In order to prevent confusion, a line segment in the vector algebra system will always be represented as the difference of 2 vectors, not as a single vector. A single vector will always represent a point.) An angle formed by 2 line segments is represented as the normalized dot* product of 2 vectors.

The common geometric relationships also have counterparts in the vector

*The dot product of 2 vectors A and B is defined to be ($|A||B|\cos\theta$) where θ is the angle between the two vectors and where $|A|$ is the magnitude of the vector A.

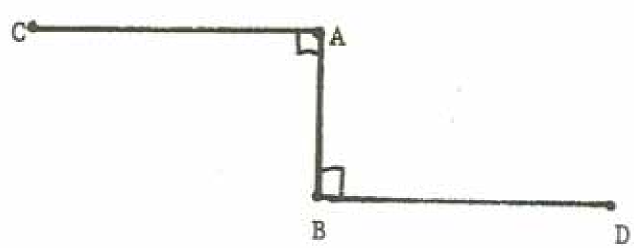
algebra system. For example, the equality of 2 line segments is represented by the equality of the magnitude of 2 vectors. A summary of the common geometric terms and relationships and their vector equivalents is given in Table I.

A discussion of the above method by representing geometric problems in terms of vector algebra problems can be found in [1].

This algebraic description of geometry has some annoying differences with geometry. First, the representation of a point as a single vector variable causes some difficulties. Although we want to consider our vectors as two dimensional, the representation of them as a single variable allows them to have any number of dimensions. That is, suppose we have a geometric point A, we represent this fact algebraically by defining a variable A. Now we would like A to be two dimensional (so our geometric and algebraic representations will be isomorphic) that is, $A = (x_1, x_2)$. But by the vector variable A could be represented as $A = (x_1, x_2, x_3)$ or $A = (x_1, x_2, \ldots, x_N)$. So although all points in geometry are coplanar, in the algebraic description of it, not all the points may be coplanar.

The result of this difference is that the geometry relation, that is, if the alternate interior angles of 2 lines are equal, then the lines are parallel is not necessarily true in the algebraic description of geometry.

<u>EXAMPLE 51</u>



COMMON GEOMETRIC TERMS AND RELATIONSHIPS WITH THEIR VECTOR ALGEBRA EQUIVALENTS

Geometric Term or Relation	Geometric Representation	Vector Algebra Representation
Point	point A	A
Line Segment	segment AB	(A − B)
Angle	angle ABC	$\frac{(A - B)(C - B)}{\lvert A - B\rvert \lvert C - B\rvert}$
Angle Equality	angle ABC = angle PQR	$\frac{(A - B)(C - B)}{\lvert A - B\rvert \lvert C - B\rvert} = \frac{(P - Q)(R - Q)}{\lvert P - Q\rvert \lvert R - Q\rvert}$
Angle Inequality	angle ABC < angle PQR	$\frac{(A - B)(C - B)}{\lvert A - B\rvert \lvert C - B\rvert} > \frac{(P - Q)(R - Q)}{\lvert P - Q\rvert \lvert R - Q\rvert}$
Colinearity	colinear A, B, C or angle ABC = 180°	colin A, B, C or $\lvert A - B\rvert + \lvert B - C\rvert = \lvert A - C\rvert$ or $(k + 1)B = kA + C$ where k is some non-zero constant or $\frac{(A - B)(C - B)}{\lvert A - B\rvert \lvert C - B\rvert} = -1$
Midpoint	M is the midpoint of AB or colinear A, M, B and AM = MB	$M = \frac{1}{2}(A + B)$
Ratio of 2 segments of a line segment	colinear A, X, B and (AX/BX) = k	$(k + 1)x = A + kB$ or $\frac{\lvert A - X\rvert}{\lvert B - X\rvert} = k$

TABLE I

COMMON GEOMETRIC TERMS AND RELATIONSHIPS WITH THEIR VECTOR ALGEBRA EQUIVALENTS

Geometric Term or Relation	Geometric Representation	Vector Algebra Representation
Non-Colinearity	non-colinear ABC or $0° <$ angle ABC $< 180°$ or $0° <$ angle ACB $< 180°$ or $0° <$ angle CAB $< 180°$ or	ncolin A, B, C or $-1 < \frac{(A - B)(C - B)}{\lvert A - B\rvert \lvert C - B\rvert} < 1$ or $-1 < \frac{(A - C)(B - C)}{\lvert A - C\rvert \lvert B - C\rvert} < 1$ or $-1 < \frac{(C - A)(B - A)}{\lvert C - A\rvert \lvert B - A\rvert} < 1$ or there does <u>not</u> exist a constant k such that $(k + 1)B = A + kC$ or $\lvert A - B\rvert + \lvert B - C\rvert > \lvert A - C\rvert$
Segment Equality	AB = CD	$\lvert A - B\rvert = \lvert C - D\rvert$
Segment Inequality	AB < CD	$\lvert A - B\rvert < \lvert C - D\rvert$
Parallel Line	AB parallel to CD	$(A - B) = k(C - D)$ where k is some non-zero constant
Triangle	triangle ABC non-colinear A, B, C	Ncolin A, B, C or $\lvert A - B\rvert + \lvert B - C\rvert > \lvert A - C\rvert$ or $\lvert A - B\rvert + \lvert A - C\rvert > \lvert B - C\rvert$ or $\lvert A - C\rvert + \lvert B - C\rvert > \lvert A - B\rvert$
Right Angle	right angle ABC or AB is perpendicular to BC	$\frac{(A - B)(C - B)}{\lvert A - B\rvert \lvert C - B\rvert} = 0$

TABLE I (continued)

In geometry AC $\perp$ AB and BD $\perp$ AB implies that AB $||$ BD.

In the algebraic description (A - C)(B - A) = 0 and (B - D)(B - A) = 0 does not imply that (A - C) = k(B - D) where k is some non-zero constant. Imagine the case where A, B, C, and D are all 3 dimensional vectors. Then if (A - C) is orthogonal to (A - B) with (A - C) a vector not in the same plane as (B - D) (i.e., (A - C) is orthogonal to the surface of the paper) then the relation (A - C) = k(B - D) does not hold.

Another difficulty with the algebraic representation is that it is not possible to represent adequately the addition of angles. In geometry the addition of angles is a very straightforward operation.

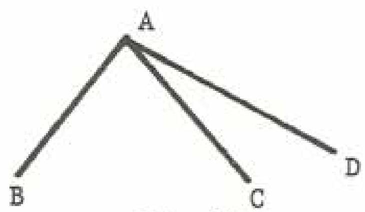


Fig.11

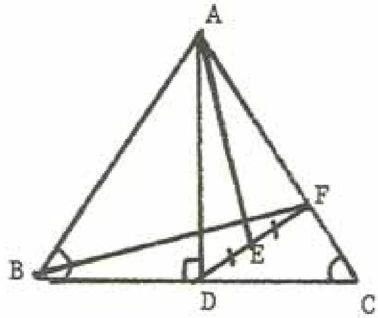
In Fig. 11(angle BAC) + angle (CAD) = angle (BAD). But in the vector algebra system $\frac{(B - A)(D - A)}{|B - A||D - A|}$ is not always equivalent to $\frac{(B - A)(C - A)}{|B - A||C - A|}$ + $\frac{(C - A)(D - A)}{|C - A||D - A|}$. This is because of the characteristics of the cosine function.

A third deficiency of the algebraic description is that due to the nature of the cosine function, the vector algebra representation of an angle is not unique for angles greater than 180 degrees.

Although these differences between geometry and this vector algebra system are annoying, none of these differences impede the solution of any of the problems discussed in this thesis.

Using the above transformation method, we can convert geometry problems into vector algebra problems. In the vector system, a geometry problem consisting of a diagram and a goal is represented by a set of simultaneous vector equations and a goal.

EXAMPLE 52



The geometric description of the problem is:

GIVEN: triangle ABC, AB = AC, AD is perpendicular to BC, DF is perpendicular to AC, E is the midpoint of DF

PROVE: AE is perpendicular to BF

Using the transformations described in Table I, the vector algebra description of the problem is

GIVEN: (1) Ncolin A, B, C

(2) $|A - B| = |A - C|$

(3) $(A - D)(B - C) = 0$

(4) $(D - F)(A - C) = 0$

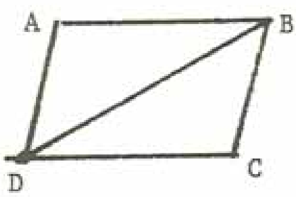
(5) $E = \frac{1}{2}(D + F)$

PROVE: $(A - E)(B - F) = 0$

3.2 The Solution of Vector Geometry Problems

In geometry the solution of a problem consists of proving a geometric relationship utilizing the problem figure and a set of basic axioms. In the vector algebra system the solution* of a problem consists of deducing a vector algebra relationship from a set of simultaneous vector equations. There are two ways to approach these vector problems. One way is to regard the problem as a geometry problem with a different kind of notation. Using this approach the method of solution is identical to the method used in geometry problems. The solutions consist of a series of geometric deductions translated into vector notation. The second way of approaching these vector geometry problems is to forget that there is a geometric interpretation of these problems and to regard the problem as one of algebraic manipulation.

EXAMPLE 53



GIVEN (1) Non-colinear A, B, C

(2) Non-colinear B, C, D

(3) AB || CD

(4) AB = CD

PROVE: AD = BC

The equivalent vector algebra problem is

GIVEN: (1) Ncolin A, B, C

(2) Ncolin B, C, D

(3) $(A - B) = k(C - D)$, k is some non-zero constant

(4) $|A - B| = |C - D|$

*In these algebraic solutions we will usually make the assumption that for any two variables A and B, $A \neq B$. This will prevent some degenerate cases from appearing in our problems. e.g., if $A = B$, what does $\frac{(A - B)(C - B)}{|A - B||C - B|}$ equal?

PROVE: $|A - D| = |B - C|$

Now let's consider 2 possible methods of solution for this vector geometry problem.

First we can consider this to be just a transformed geometry problem. From (3) we can deduce that

$$(5)\ \frac{(A - B)(D - B)}{|A - B|\,|D - B|} = \frac{(C - D)(B - D)}{|C - D|\,|B - D|}$$

By identity we get

$$(6)\ |B - D| = |B - D|$$

So using (1), (2), (4), (5), and (6) we have obtained a vector version of the SIDE-ANGLE-SIDE congruent triangle. Therefore we can deduce that $|A - D| = |B - C|$ by the equivalence of corresponding parts of congruent triangles.

Now we will consider the problem to be one of algebraic manipulation. By definition $|A - B|$ = Square root $((A - B)^2)$. So using (3) and (4) we get

$$(A - B)^2 = k^2(C - D)^2$$

$$(A - B)^2 = (C - D)^2 \text{ so } k^2 = 1$$

therefore

$$(5)\ (A - B) = (C - D) \text{ or } (D - C)$$

Rearranging (5) we get

$$(5')\ (A - D) = (B - C) \text{ or } (C - B)$$

$$(5'')\ (A - D)^2 = (B - C)^2 \text{ or } (C - B)^2$$

$$(5''')\ |A - D|^2 = |B - C|^2$$

$$(5'''')\ |A - D| = |B - C|$$

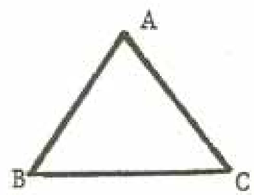
Now in the next few sections we will look at in more depth the 2 approaches to the solutions of the vector geometry problems.

3.2.1 Solution of Vector Geometry Problems by Algebraic Manipulation

Vector geometry problems can be divided into 2 main classes when considered from the point of view of solution by algebraic manipulation. The 2 classes are those problems which do not require the use of a non-colinearity constraint and those problems which do require a non-colinearity constraint

EXAMPLE 54

To illustrate the use and the lack of use of the non-colinearity constraint let us examine one of the simplest geometry problems, the isosceles triangle theorem (pons asinorum)



The geometric description of this problem is

GIVEN: (1) non-colinear A, B, C

(2a) AB = AC

(2b) angle ABC = angle BCA

PROVE: (a) angle ABC = angle BCA

(b) AB = AC

The vector algebra description of the problem is

GIVEN: (1) ncolin A, B, C

(2a) $|A - B| = |A - C|$

(2b) $\frac{(A - B)(C - B)}{|A - B||C - B|} = \frac{(B - C)(A - C)}{|B - C||A - C|}$

PROVE: (a) $\frac{(A - B)(C - B)}{|A - B||C - B|} = \frac{(B - C)(A - C)}{|B - C||A - C|}$

(b) $|A - B| = |A - C|$

Now let's look at the solution of part (a). We must prove

$$\frac{(A - B)(C - B)}{|A - B||C - B|} = \frac{(B - C)(A - C)}{|B - C||A - C|}$$

By identity we get

(3) $|B - C| = |B - C|$

We can reduce the goal to proving

$$(A - B)(C - B) = (B - C)(A - C)$$

(4) $AC - AB - BC + B^2 = AB - BC - AC + C^2$

From (2a) we get

(2a) $|A - B| = |A - C|$

(2a') $A^2 + B^2 - 2AB = A^2 + C^2 - 2AC$

Rearranging terms we get

$$AC - AB + B^2 = AB - AC + C^2$$

which is a form of (4) so the problem is solved.

Note that we only used constraint (2a) to obtain a solution. The non-colinearity constraint of (1) was never needed.

Now let's look at the solution of part (b). We must prove

$$|A - B| = |A - C|$$

A solution cannot be obtained without the non-colinearity constraint of (1).

Fig. 12

In Fig. 12 $\frac{(A - B)(C - B)}{|A - B||C - B|} = \frac{(B - C)(A - C)}{|B - C||A - C|} = 0$, but it is not necessarily true that $|A - B| = |A - C|$

Now we will show how to obtain a solution using the non-colinearity constraint. The constraint in (1) will be interpreted as

$$(1') \quad -1 < \frac{(B - A)(C - A)}{|B - A||C - A|} = (\text{ang } A) < 1$$

(2b) can be written as

$$(2b') \quad \frac{|A - B| - |A - C|(\text{ang } A)}{|B - C|} = \frac{|A - C| - |A - B|(\text{ang } A)}{|B - C|}$$

or

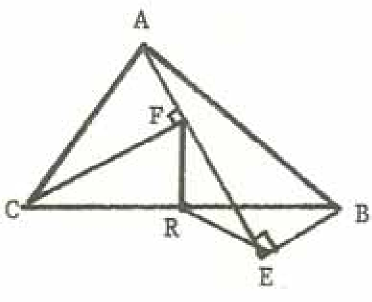
$$(2b'') \quad (1 + (\text{ang } A))|A - B| = (1 + (\text{ang } A))|A - C|$$

since $-1 < (\text{ang } A) < 1$ we can divide both sides of (2b") by $(1 + \text{ang } A))$ and get

$$|A - B| = |A - C| .$$

The solution of problems which do not require the non-colinearity constraint is usually quite straightforward. Most solutions involve only simple substitutions and the creation of linear combinations of the constraint equations.

<u>EXAMPLE 55</u>, (2)*



*In some vector algebra examples, a number in parenthesis will be given. This number will refer to a previous example where the problem was described in geometric terms.

GIVEN: (1) $(C - F)(F - E) = 0$

(2) $(B - E)(F - E) = 0$

(3) $R = \frac{1}{2}(C + B)$

(4) colin C, D, B

(5) ncolin A, B, C

PROVE: $|R - F| = |R - E|$

If this problem were to be solved geometrically, a construction would have to be drawn. In the vector algebra system, the solution does not require a construction. Also the non-colinearity constraint of (5) is not required in the vector solution.

SOLUTION:

(1) $(C - F)(F - E) = CF - CE - F^2 + EF = 0$

(2) $(B - E)(F - E) = BF - BE - EF + E^2 = 0$

The goal is $|R - F| = |R - E|$ or

$$R^2 + F^2 - 2FR = R^2 + E^2 - 2ER \quad \text{or}$$

$$F^2 - 2FR = E^2 - 2ER$$

Now substituting (3) the goal becomes

(4) $F^2 - CF - BF = E^2 - CE - BE$

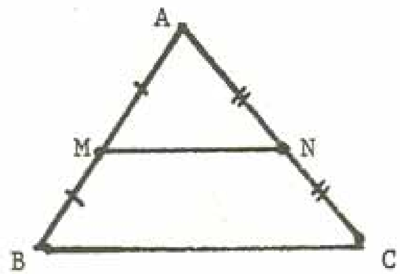
We can easily derive (4) by (2) - (1)

$$BF - BE - EF + E^2 = -CF + CE - F^2 - EF$$

$$E^2 - CE - BE = F^2 - CF - BF$$

Q.E.D.

EXAMPLE 56, (1)



GIVEN: (1) $M = \frac{1}{2}(A + B)$

(2) $N = \frac{1}{2}(A + C)$

(3) ncolin A, B, C

PROVE: $(M - N) = \frac{1}{2}(B - C)$

The non-linearity constraint of (3) will not be required in the solution.

SOLUTION: (1) - (2) gives

$(M - N) = \frac{1}{2}(A + B - A - C)$ or

$(M - N) = \frac{1}{2}(B - C)$

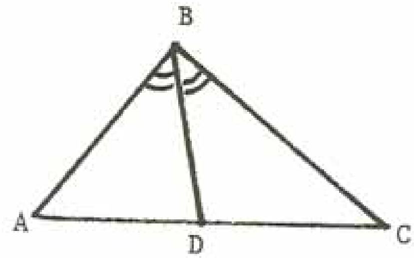
Q.E.D.

The solution of vector geometry problems which do require the use of a non-colinearity constraint in a problem is quite complicated.

There are several reasons for the great complexity of these non-colinearity constraint (NCC) problems. The main reason is that many of the more difficult problems in geometry (e.g., the internal angle bisector problem and the Theorem of Morley) require the use of a non-colinearity constraint. Another reason is that there are a number of ways to utilize the non-colinearity constraint when solving a problem.

EXAMPLE 57, (26)

These next 3 examples will illustrate 3 ways in which the non-colinearity constraint can be used to solve a vector geometry problem. The algebraic derivations that will be given in these next examples will generally be very sketchy. The interested (and masochistic) reader may try to fill in the details himself.



GIVEN: (1) $\frac{(A - B)(B - D)}{|A - B||B - D|} = \frac{(D - B)(B - C)}{|D - B||B - C|}$

(2) colin A, D, C or $(k_1 + 1)D = A + k_1 C$

(3) ncolin A, B, C or $-1 < \frac{(A - B)(C - B)}{|A - B||C - B|} < 1$

Let $k_2 = \frac{|A - B|}{|B - C|}$

PROVE: $\frac{|A - B|}{|B - C|} = k_2 = \frac{|A - D|}{|D - C|} = k_1$

Into (1) we substitute for D using (2)(and with some rearranging)

$$\frac{AB - AD - B^2 + BD}{BD - CD - B^2 + BC} = k_2$$

Let $\frac{(A - B)(C - B)}{|A - B||C - B|} = (\text{ang } B)$

$$\frac{\frac{k_1}{k_1+1}(\text{ang } B) + \frac{k_2}{k_1+1}}{\frac{1}{k_1+1}(\text{ang } B) + \frac{k_1}{k_2(k_2+1)}} = k_2$$

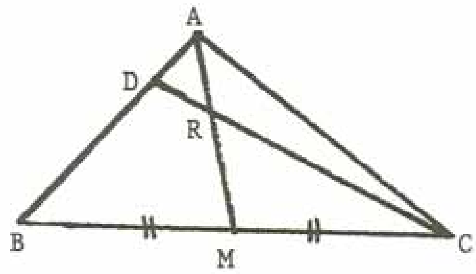
$$k_1((\text{ang } B) - 1) = k_2((\text{ang } B) - 1)$$

so since $-1 < (\text{ang } B) < 1$, we can say that $k_1 = k_2$

Q.E.D.

<u>EXAMPLE 58</u>, (7)

This use of the non-colinearity constraint will exploit the fact that if there are 3 non-colinear vectors, every vector (which can be expressed as a linear combination of these 3 vectors) has a unique representation in terms of these 3 vectors.



GIVEN: (1) $3D = 2A + B$

(2) $M = \frac{1}{2}B + \frac{1}{2}C$

(3) colin A, R, M or $(k_1 + 1)R = k_1 A + M$

(4) colin D, R, C or $(k_2 + 1)R = k_2 D + C$

(5) ncolin A, B, C

PROVE: $\frac{|R - M|}{|A - R|} = k_1 = 1$

Substituting (1) into (4) and (2) into (3) and rearranging we get

$$(3')\ R = \frac{k_1}{k_1+1} A + \frac{1}{k_1+1} (\tfrac{1}{2}B + \tfrac{1}{2}C)$$

$$(4')\ R = \frac{k_2}{k_2+1} (\frac{2}{3} A + \frac{1}{3} B) + \frac{1}{k_2+1} C$$

Now by exploiting the non-colinearity of A, B, and C we can equate the constants for each of them in (3') and (4'). So

$$\frac{1}{2}\,\frac{1}{k_1+1} = \frac{\frac{1}{3}\,k_2}{k_2+1}$$

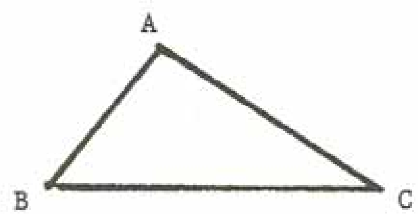
$$\frac{k_1}{k_1+1} = \frac{\frac{2}{3}\,k_2}{k_2+1}$$

$$\frac{\frac{1}{2}}{k_1+1} = \frac{1}{k_2+1}$$

From the above 3 equations we can deduce that $k_1 = \frac{|R - M|}{|A - R|} = 1.$

Q.E.D.

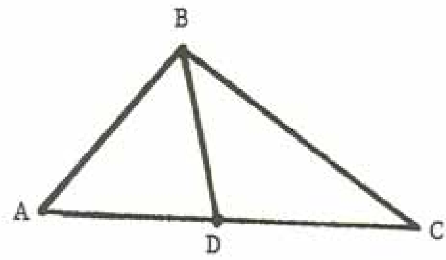
<u>EXAMPLE 59</u>, (26)



This use of the non-colinearity constraint will utilize a vector version of the Law of Sines. The algebraic version of it is the identity that

$$\text{(I)}\quad \frac{1 - \left(\frac{(B - A)(C - A)}{|B - A|\,|C - A|}\right)^2}{|B - C|^2} = \frac{1 - \left(\frac{(A - B)(C - B)}{|A - B|\,|C - B|}\right)^2}{|A - C|^2}$$

The above identity is really the vector algebra version of the law of sine[2].



GIVEN: (1) $\frac{(A - B)(B - D)}{|A - B||B - D|} = \frac{(D - B)(B - C)}{|D - B||B - C|}$

(2) colin A, D, C or $(k_1+1)D = A + k_1C$

(3) ncolin A, B, C or $-1 < \frac{(A - B)(C - B)}{|A - B||C - B|} < 1$

PROVE: $\frac{|A - B|}{|B - C|} = \frac{|A - D|}{|D - C|}$

To verify this goal using the identity in (I), derive the fact that

$$\frac{(A - D)(B - D)}{|A - D||B - D|} = \frac{(B - D)(C - D)}{|B - D||C - D|}$$

Then use (I).

3.2.2 The Use of Constructions in Solving Vector Geometry Problems by Algebraic Manipulation

Constructions (a construction in the vector algebra system corresponds to the introduction of a new variable and some new constraint equations) to solve vector geometry problems are not used as frequently as geometry constructions are used. The reader may have already noticed that many of the problems that required a construction when solved geometrically did not require a construction when solved by algebraic manipulation. This is due to the form of some of the constructions. For example, suppose we construct a midpoint M of a line AB in the geometric solution of a problem. Then we can usually formulate a vector algebra proof of the problem without a construction. This can be done by translating the geometry proof into vector algebra terms.(See the next section for an example of translating geometry methods into vector algebra terms. In the section the 5 geometry congruency theorems are translated in vector algebra terms.) Now the midpoint M we constructed for the geometric solution when translated in vector algebra terms becomes equal to the expression $\frac{1}{2}(A + B)$. So we can express the new variable M in terms of previously existing ones. So it is not necessary to introduce the new variable M into the vector version of the proof. We just replace all instances of it by an equivalent expression of previously defined variables. Therefore we have a vector algebra proof of the problem which does not require a construction (i.e., introduction of a new variable). The problems which can be solved geometrically using the midpoint reflection construction, SCH2, or SCH4 can usually be solved algebraically without a construction.

This method of elimination of the need for a construction is not always useful. Suppose we have a geometric construction to draw a perpendicular AD from a point A to line BC. In vector algebra terms we would introduce a variable D such that $(A - D)(B - C) = 0$. Expressing D in terms of previously defined variables is quite difficult.

There are a significant number of geometry problems which we believe require a construction in their algebraic solution. (Due to the many methods of algebraic solution, we cannot definitely decide which problems require an algebraic solution. In this paper we will use the criteria that if a problem has no straightforward algebraic solution then it will be considered to require a construction. By a straightforward solution it is meant the methods used to solve the problems in this chapter.)

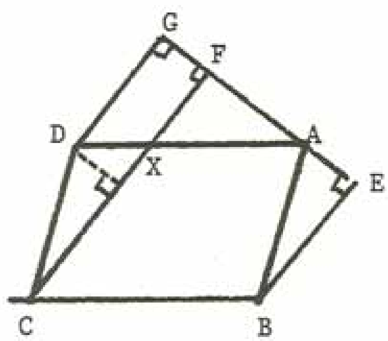
There is one algebraic construction heuristic which is useful.

Vector Construction Heuristic 1 (VCH1)

If the goal of the problem is to prove an equality of the form $|A - B| + |C - D| = |P - Q|$, define a new variable X such that $|P - X| + |Q - X| = |P - Q|$ and $|P - X| = |A - B|$. This construction has the effect of reducing the goal to proving $|C - D| = |Q - X|$.

The motivation for this algebraic construction is that all the algebraic problem solving methods are easy to apply to problems that involve proving the magnitude of 2 vectors are equal. It is not very easy to try to prove an equality about the sum of the magnitudes of 2 vectors. This construction is able to convert the problem so that the algebraic methods can be more easily applied.

EXAMPLE 60, (17)



GIVEN: (1) $(D - A) = (C - B)$

(2) $(C - F)(E - G) = 0$

(3) $(D - G)(E - G) = 0$

(4) $(B - E)(E - G) = 0$

(5) Colin G, F, A, E

PROVE: $|C - F| = |B - E| + |D - G|$

By defining X so that $|D - G| = |X - F|$ we can reduce the goal to proving $|C - X| = |B - E|$. This can be done using straightforward methods.

This algebraic heuristic is very similar to the geometry construction heuristic SCH1. This is due to the fact that both the geometric and albebraic methods are not convenient for proving equality about a sum of segments or vector magnitudes. Both methods are geared for proving equality between a single pair of elements.

This algebraic heuristic is the only one which has been formulated. In section 3.3.1 we present a vector version of some of the other geometry construction heuristics presented earlier. These heuristics cannot really be considered to be algebraic heuristics since there is really no algebraic motivation for them. Their motivation really has a geometric nature. We considered VCH1 to be an algebraic heuristic because it can be given a good algebraic motivation. SCH1 and VCH1 are similar because the geometric and algebraic problem solving methods share some common properties.

There are other geometry problems which require an algebraic construction in their algebraic solution. These geometry problems are mostly those which can be solved geometrically using SCH3 and SCH6. We will not present any algebraic construction heuristics for these problems. Due to the complexity of achieving a purely algebraic solution for these problems, we have been unable to analyze adequately these problems from an algebraic point of view. As stated earlier, a vector version of SCH3 or SCH6 is not really a satisfactory algebraic construction heuristic since the heuristics have no real algebraic motivation.

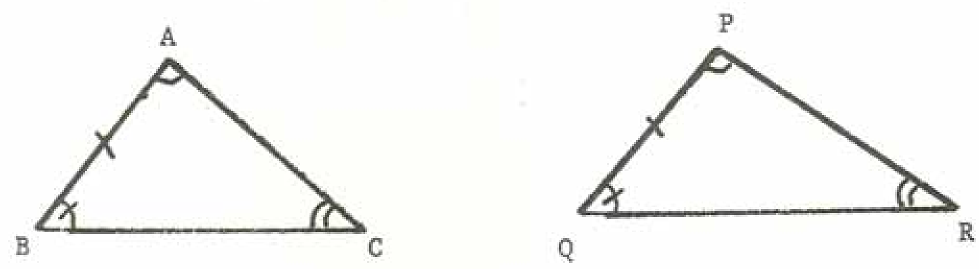
To conclude this section we will summarize its results. Geometry problems solved by using reflection, SCH2 or SCH4 generally do not require an algebraic construction. Geometry problems solved by SCH1, SCH3 and SCH6 for the most part require an algebraic construction. The only algebraic construction heuristic that has been formulated has been for the problems solved by SCH1.

3.2.3 A Relation Between the Algebraic and Geometric Solutions of a Problem

The use or lack of use of the non-colinearity constraint in a vector geometry problem can also be related to the solution of the corresponding geometry problem.

The main methods of proof in geometry are the 5 congruent triangle theorems. (The similar triangle theorem is only a slight generalization of the Side-Angle-Angle and Angle-Side-Angle congruency theorems and will not be discussed.) In examples 57 - 61 we present algebraic derivations of the 5 congruent triangle theorems. Of the 5 theorems, only the Side-Angle-Angle and the Angle-Side-Angle congruency theorems required the use of the non-colinearity constraint in their derivations. So any geometry problem which can be solved using only the other 3 kinds of congruency theorems will not require a non-colinearity constraint for its algebraic solution. Only those problems which use the Side-Angle-Angle or the Angle-Side-Angle theorems will require a non-colinearity constraint.

<u>EXAMPLE 61</u> (Vector Version of the ASA Congruency Theorem)



GIVEN: (1) $|A - B| = |P - Q|$

(2) $$\frac{(B - A)(C - A)}{|B - A||C - A|} = \frac{(Q - P)(R - P)}{|Q - P||R - P|}$$

(3) $$\frac{(A - B)(C - B)}{|A - B||C - B|} = \frac{(P - Q)(R - Q)}{|P - Q||R - Q|}$$

(4) Ncolin (A, B, C), Ncolin (P, Q, R)

PROVE: $|A - C| = |P - R|$

SOLUTION:

Using (1), rew ite (2) and (3) as

(2') $$\frac{(B - A)(C - A)}{|C - A|} = \frac{(Q - P)(R - P)}{|R - P|}$$

(3') $$\frac{(A - B)(C - B)}{|C - B|} = \frac{(P - Q)(R - Q)}{|R - Q|}$$

In (2') substitute $(B - A)(C - A) = |A - C|^2 - (B - C)(A - C)$

and $(Q - P)(R - P) = |P - R|^2 - (Q - R)(P - R)$

$$\frac{|A - C|^2 - (B - C)(A - C)}{|A - C|} = \frac{|P - R|^2 - (Q - R)(P - R)}{|P - R|}$$

or (2")

$$|A - C| - |B - C| \frac{(B - C)(A - C)}{|A - C||B - C|}$$

$$= |P - R| - |Q - R| \frac{(Q - R)(P - R)}{|Q - R||P - R|}$$

In (3') substitute $(A - B)(C - B) = |C - B|^2 - (B - C)(A - C)$

and $(P - Q)(R - Q) = |R - Q|^2 - (Q - R)(P - R)$

(3") $$|C - B| - |A - C| \frac{(B - C)(A - C)}{|A - C||B - C|}$$

$$= |R - Q| - |P - R| \frac{(Q - R)(P - R)}{|Q - R||P - R|}$$

Now we will use the vector algebra version of the geometry that if 2 angles of a triangle are equal to 2 angles of another triangle, the third angle of each triangle is equal. So by using this theorem we can say

$$\frac{(B - C)(A - C)}{|A - C||B - C|} = \frac{(Q - R)(P - R)}{|Q - R||P - R|} = k$$

so

$$(2'')\ \ |A - C| - k|B - C| = |P - R| - k|Q - R|$$

$$(3'')\ \ |B - C| - k|A - C| = |Q - R| - k|P - R|$$

Computing (2") + k(3") we get

$$(5)\ \ (1 - k^2)|A - C| = (1 - k^2)|P - R|$$

Now we will invoke the non-colinearity constraint in (4). Since ncolin (A, B, C), we can say that

$$-1 < \frac{(B - C)(A - C)}{|B - C||A - C|} = k < 1$$

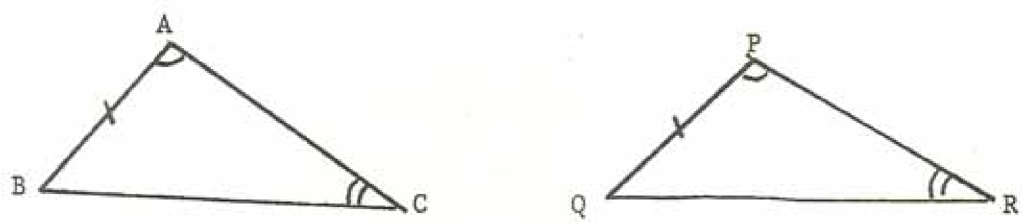
This allows us to derive from (5) that

$$|A - C| = |P - R|$$

Q.E.D.

Notice that in this problem the non-colinearity constraint was necessary for the solution of the problem.

<u>EXAMPLE 62</u> (Vector Version of the SAA congruency theorem)



GIVEN: (1) $|A - B| = |P - Q|$

(2) $\frac{(B - A)(C - A)}{|B - A||C - A|} = \frac{(Q - P)(R - P)}{|Q - P||R - P|}$

(3) $\frac{(A - C)(B - C)}{|A - C||B - C|} = \frac{(P - R)(Q - P)}{|P - R||Q - P|}$

(4) Ncolin (A, B, C), Ncolin (P, Q, R)

PROVE: $|B - C| = |Q - R|$

SOLUTION:

Use the vector algebra version of the geometry theorem that if 2 angles of a triangle are equal to 2 angles of another triangle, the third angle of each triangle is equal. This will allow us to say that

$$\frac{(A - B)(C - B)}{|A - B||C - B|} = \frac{(P - Q)(R - Q)}{|P - Q||R - Q|}$$

Now use the same derivation we used to derive the ASA theorem.

Q.E.D.

To show that the ASA and SAA theorems always require the non-colinearity constraint we use the following diagram.

Now in the above diagrams A, B, and C, and also P, R, and R are colinear. Also

$$|B - C| = |Q - R|$$

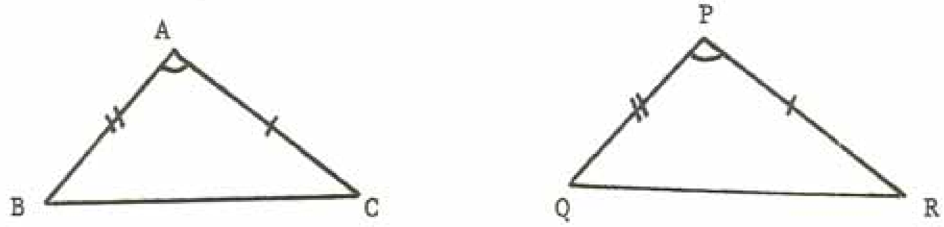
$$\frac{(B - C)(A - C)}{|B - C||A - C|} = \frac{(Q - R)(P - R)}{|Q - R||P - R|}$$

$$\frac{(C - A)(B - A)}{|C - A||B - A|} = \frac{(R - P)(Q - P)}{|R - P||Q - P|}$$

$$\frac{(C - B)(A - B)}{|C - B||A - B|} = \frac{(R - Q)(P - Q)}{|R - Q||P - Q|}$$

The conditions of both the ASA and SAA theorems are satisfied (Except for the non-colinearity constraint). But $|A - C|$ is not necessarily equal to $|P - R|$. So both theorems must require the non-colinearity constraint.

<u>EXAMPLE 63</u> (Vector Version of the SAS congruency theorem)



GIVEN: (1) $|A - B| = |P - Q|$

(2) $|A - C| = |P - R|$

(3) $\frac{(B - A)(C - A)}{|B - A||C - A|} = \frac{(Q - P)(R - P)}{|Q - P||R - P|}$

PROVE: $|B - C| = |Q - R|$

From (1), (2), and (3)

$$(B - A)(C - A) = (Q - P)(R - P)$$

(4) $BC - AB - AC + A^2 = QR - PQ - PR + P^2$

From (1)

(5) $A^2 + B^2 - 2AB = P^2 + Q^2 - 2PQ$

From (2)

(6) $A^2 + C^2 - 2AC = P^2 + R^2 - 2PR$

Compute (5) + (6) - 2(4)

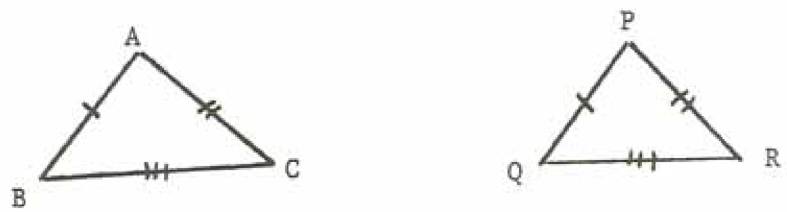
$$B^2 + C^2 - 2BC = Q^2 + R^2 - 2QR$$

$$|B - C|^2 = |P - Q|^2$$

$$|B - C| = |P - Q|$$

Q.E.D.

<u>EXAMPLE 64</u> (Vector Version of the SSS congruency theorem)



GIVEN: (1) $|A - B| = |P - Q|$

(2) $|A - C| = |P - R|$

(3) $|B - C| = |Q - R|$

PROVE: $\dfrac{(B - A)(C - A)}{|B - A||C - A|} = \dfrac{(Q - P)(R - P)}{|Q - P||R - P|}$

Using (1) and (2) the goal can be changed to

$$(B - A)(C - A) = (Q - P)(R - P)$$

$$BC - AB - AC + A^2 = QR - PQ - PR + P^2$$

Using (1), (2), (3)

(1') $A^2 + B^2 - 2AB = P^2 + Q^2 - 2PQ$

(2') $A^2 + C^2 - 2AC = P^2 + R^2 - 2PR$

(3') $B^2 + C^2 - 2BC = Q^2 + R^2 - 2QR$

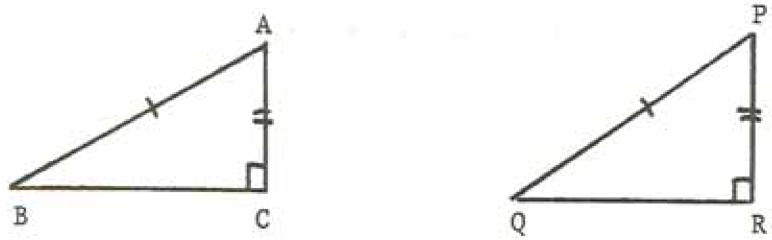
Compute (1') + (2') + (3')

$$2A^2 + 2AB - 2AC + 2BC = 2P^2 - 2PQ - 2PQ + 2QR$$

$$BC - AB - 2AC + A^2 = QR - PQ - PR + P^2$$

Q.E.D.

<u>EXAMPLE 65</u> (Vector Version of the Hypothenuse-Leg congruency theorem)



GIVEN: (1) $|A - B| = |P - Q|$

(2) $|A - C| = |P - R|$

(3) $(A - C)(B - C) = (P - R)(Q - R) = 0$

PROVE: $|B - C| = |Q - R|$

From (3)

$$AB - AC - BC + C^2 = PQ - PR - QR + R^2 = 0$$

From (1)

$$A^2 + B^2 - 2AB = P^2 + Q^2 - 2PQ$$

From (2)

$$A^2 + C^2 - 2AC = P^2 + R^2 - 2PR$$

Compute (1) - (2) + 2(3)

$$B^2 - C^2 - 2AB + 2AC + 2AB - 2BC + 2C^2 - 2AC$$

$$= Q^2 - R^2 - 2PQ + 2PR + 2PQ - 2PR - 2QR + 2R^2$$

$$B^2 + C^2 - 2BC = Q^2 + R^2 - 2QR$$

$$|B - C|^2 = |Q - R|^2$$

$$|B - C| = |Q - R|$$

Q.E.D.

3.2.4 Solution of Vector Geometry Problems Using Geometric Methods

The solution of vector geometry problems using geometric methods is quite simple. This method merely regards the vector geometry problems as geometry problems expressed in a different notation.

Since this method is parallel to the geometric method of solving problems, constructions will often be needed. The next section will discuss what kind of constructions should be made in the vector algebra system. Since our method of solving the vector algebra problem is similar to that of geometry, our vector algebra constructions will also be similar to those in geometry.

3.2.4.1 Vector Algebra Description of Geometry Construction Heuristics

This section contains heuristics for generating constructions in the vector algebra representation of geometry. These heuristics are essentially a vector algebra translation of some of the situational construction heuristics described in the first half of this thesis.

VECTOR ALGEBRA VERSION OF SOME SITUATIONAL CONSTRUCTION HEURISTICS

VECTOR VERSION OF SCH2

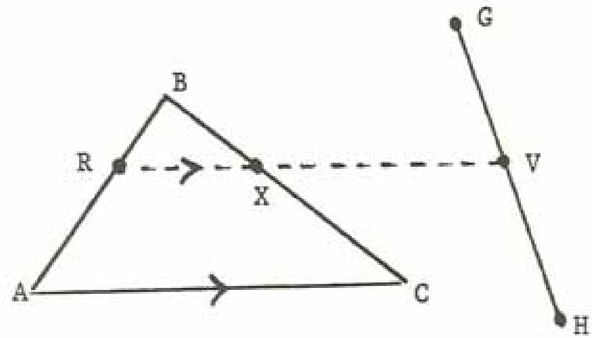
Definition: R is a ratio vector if an equation of the form $(k + 1)R = A + kB$, $k > 0$, is either a constraint of the problem or a goal of the problem. Another form of this equation could be Colin(A, R, B), $\frac{|A - R|}{|B - R|} = k$.

Situation: R is a ratio vector

Goal: to deduce some relationship

Construction: Select some vector of the form $(A - C)$. Define a new vector X to be $X = k(A - C) + R$, $k > 1$, and Colin (B, X, C). Also it may be necessary to define other vectors V, such that $V = k'(A - C) + R$ and Colin (G, V, H), where $k' > 0$ and G and

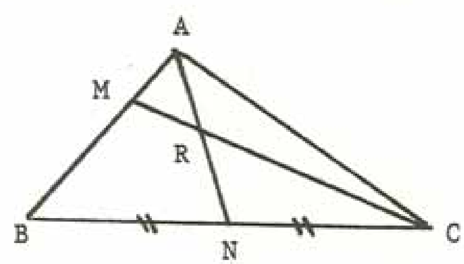
and H are 2 other vectors in the problem.



If the goal involves deducing a relationship concerning vector magnitudes or vector equalities, (i.e., if the goal is of the form $|X - Y| = |W - Z|$ or $|X - Y| < |W - Z|$ or $(X - Y) = k(W - Z)$, $k \neq 0$) try to select the vector $(A - C)$ so that it is one of the vectors involved in the goal.

EXAMPLE OF SCH2

EXAMPLE 66, (25)



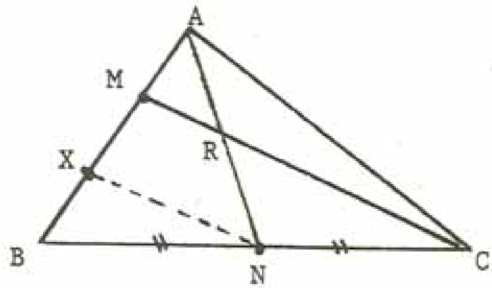
GIVEN: (1) $N = \frac{1}{2}(B + C)$ or $2N = B + C$

(2) $4M = B + 3A$

(3) Colin (A, R, N)

PROVE: $R = \frac{1}{2}(A + N)$

Select N as a ratio vector. Select $(M - C)$. Define a new vector X to be $X = k(M - C) + N$, $k > 0$, and Colin (M, X, B).



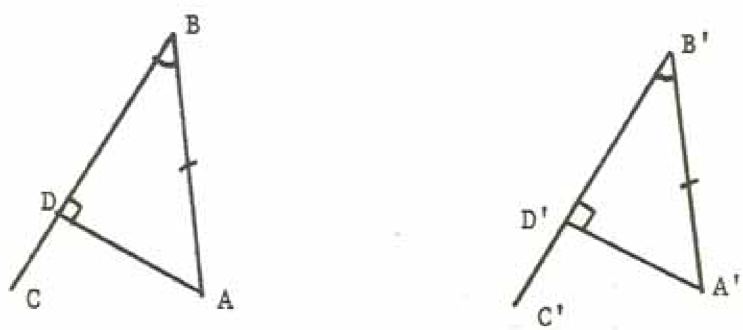
VECTOR VERSION OF SCH3

Situation: $|A - B| = |A' - B'|$, $\frac{(A - B)(C - B)}{|A - B||C - B|} = \frac{(A' - B')(C' - B')}{|A' - B'||C' - B'|}$

Goal: to prove a vector magnitude or a dot product equality

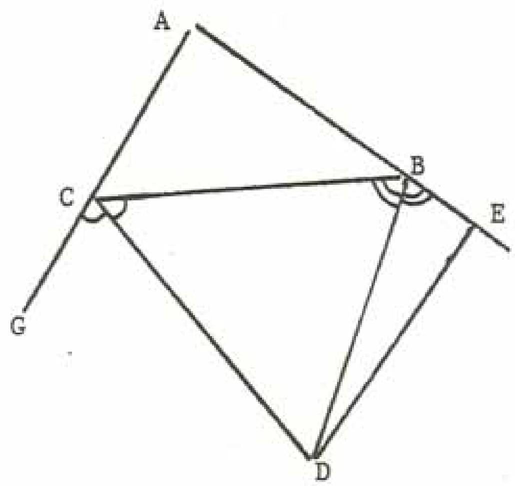
Construction: Define D to be $(A - D)(B - C) = 0$ and Colin (A, D, C).

Define D' to be $(A' - D')(B' - C') = 0$ and Colin (B', D', C')



EXAMPLE OF SCH3

EXAMPLE 67, (33)



GIVEN: (1) $\frac{(G - C)(D - C)}{|G - C||D - C|} = \frac{(D - C)(B - C)}{|D - C||B - C|}$

(2) $\frac{(C - B)(D - B)}{|C - B||D - B|} = \frac{(D - B)(E - B)}{|D - B||E - B|}$

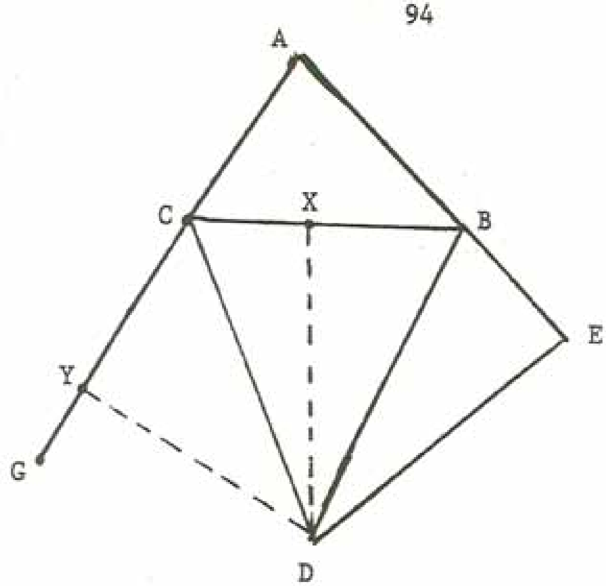
(3) $(D - E)(A - E) = 0$

(4) Colin (A, B, E)

(5) Colin (G, C, A)

PROVE: $|A - E| = \frac{1}{2}(|A - C| + |B - C| + |A - B|)$

Since the goal involves a vector magnitude equality we can use SCH3. There are 2 situations in which SCH3 can be applied. The first situation is (1) and $|D - C| = |D - C|$. The second is (2) and $|D - B| = |D - B|$. So using SCH3 we will define 2 vectors X and Y. X is a vector such that $(D - X)(B - C) = 0$ and Colin (C, X, B). Y is a vector such that $(D - Y)(G - C) = 0$ and Colin (G, Y, C).



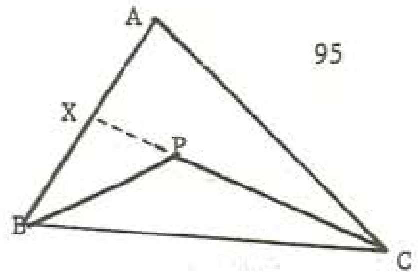
VECTOR VERSION OF SCH4

Definition: P is inside A, B, and C if Ncolin (A, B, C) and P is a vector such that if X, Y, and Z are A, B, and C or B, A, and C or C, B, and A, then there exists a vector W such that Colin (X, P, W), Colin (Y, W, Z), $|Y - W| < |Y - Z|$ and $|W - Z| < |Y - Z|$. In geometric terms these conditions state that P is a point within a triangle ABC.

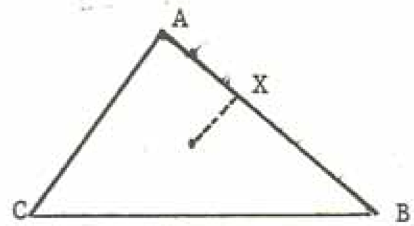
Situation: P is inside A, B, and C

Goal: to prove a vector magnitude inequality or a normalized dot product inequality

Construction: Either: (1) Choose a vector Q in the figure. Define a new vector X to be Colin (P, Q, X) and Colin (D, E, X) where D and E are either A, B, or C.



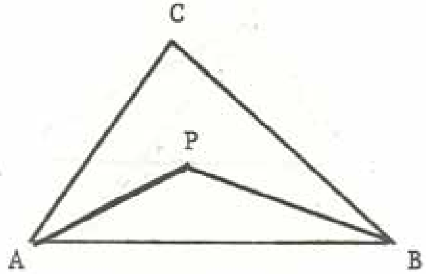
In the above figure Q = C, D = A, and E = B. Or: (2) Define a new vector X to be Colin (D, X, E) and (F - G) = k(P - X), k ≠ 0, where D, E, F, and G are either A, B, or C.



The vector algebra definition of a point being inside a triangle may seem very clumsy and difficult to use. This is because most people utilize a very intuitive concept of "a point inside a triangle". Any rigorous definition of this concept will probably be very clumsy and difficult to use.

EXAMPLES OF SCH4

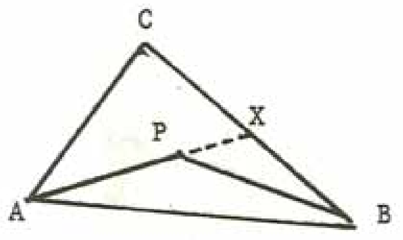
<u>EXAMPLE 68</u>, (37)



GIVEN: P is inside A, B, and C

PROVE: $\frac{(A - D)(B - D)}{|A - D||B - D|} < \frac{(A - C)(B - C)}{|A - C||B - C|}$

Since the goal involves a normalized dot product inequality we can apply SCH4. Define X such that Colin (A, P, X) and Colin (C, X, B).



VECTOR VERSION OF SCH5

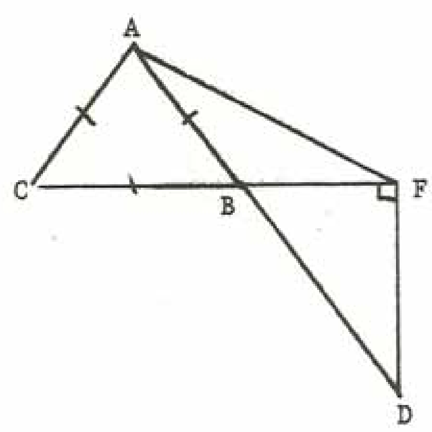
<u>Situation</u>: $|A - B| = 2|C - D|$ and the vector $X = \frac{1}{2}(A + B)$ is not defined in the prove

<u>Goal</u>: to prove anything

<u>Construction</u>: Define the vector $X = \frac{1}{2}(A + B)$. Then if needed the other construction heuristics can be applied.

EXAMPLES OF SCH5

<u>EXAMPLE 69</u>, (41)



GIVEN: (1) $|A - C| = |A - B| = |B - C|$

(2) $|B - D| = 2|A - B|$

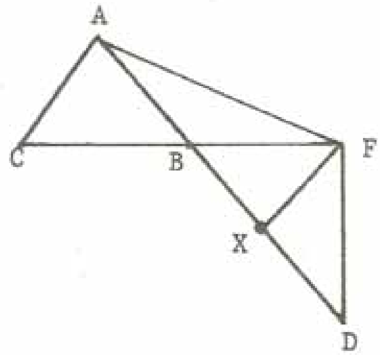
(3) $(F - D)(F - C) = 0$

(4) Colin (A, B, D)

(5) Colin (C, B, F)

PROVE: $(F - A)(C - A) = 0$

Since we have (2), we can use SCH5 and define $X = \frac{1}{2}(B + D)$

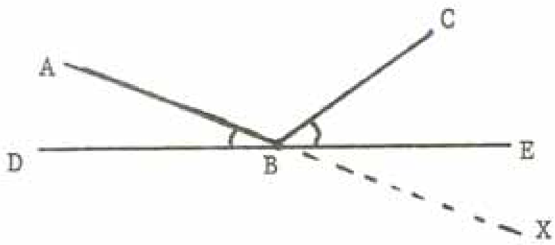


VECTOR VERSION OF SCH6

<u>Situation</u>: $\frac{(A - B)(D - B)}{|A - B||D - B|} = \frac{(C - B)(E - B)}{|C - B||E - B|}$, Colin (D, B, E), and for all vectors V, if Colin (A, V, C), $|V - A| < |A - C|$, and $|V - C| < |A - C|$, then Ncolin (D, V, E). This last condition in geometric terms states that A and C should be on the same side of the line DE.

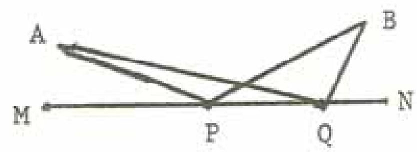
<u>Goal</u>: to prove an equality or inequality with one of the terms being the sum $(|A = B| + |B - C|)$. E.G., $|A - B| + |B - C| = |P - Q|$.

<u>Construction</u>: Define X such that $|B - X| = |B - C|$, Colin (A, B, X), $|B - X| < |A - X|$, and



EXAMPLE OF SCH6

EXAMPLE 70, (42)



GIVEN: (1) For all vectors V, if Colin (A, V, B), $|V - A| < |A - B|$ and $|V - B| < |A - B|$, then Ncolin (M, V, N)

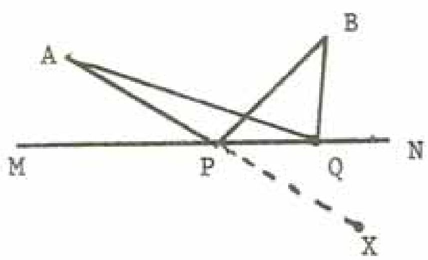
(2) $\frac{(A - P)(M - P)}{|A - P||M - P|} = \frac{(B - P)(N - P)}{|B - P||N - P|}$

(3) Colin (M, P, Q, N)

(4) $P \neq Q$

PROVE: $(|A - P| + |B - P|) < (|A - Q| + |B - Q|)$

Since we have (1), (2), and (3) we can use SCH6 and define X such that $|P - X| = |P - B|$, Colin (A, P, X), and $|P - X| < |A - X|$.



4 SUGGESTIONS FOR FUTURE WORK

There are several areas in which future work could be done.

One area would be the field of geometry theorem proving. Work would involve the incorporating the geometry construction heuristics of this thesis into the framework of a geometry theorem prover (such as the one described in [2]). This type of work would first require a set of heuristics to decide when to attempt to draw a construction during the course of solving a geometry problem.

Another area of future work could be to explore the use of a symbol manipulation system (such as the one described in [3]) to help solve the vector algebra versions of geometry problems. As was noted previously, some of the algebraic manipulation necessary to solve the vector geometry problems is quite formidable. If the use of the symbol manipulation system was successful, this would provide a new kind of practical geometry theorem prover.

In other branches of mathematics there are operations similar to that of the geometric construction. For example, in group theory a construction operation could correspond to the insertion of a term of the form $(a\ a^{-1})$ into an expression so that it could be more easily evaluated. This kind of construction is used to help solve word problems for groups (see [4] for a discussion of the word problem).

For future work it would be useful to formulate heuristics for the use of the construction operation in other branches of mathematics. Comparisons could then be made between the heuristics for construction operations in various branches of mathematics. These comparisons could then be used to identify some general properties of construction heuristics.

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Problems used in the examples were taken from:

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